# Probability in the Analysis of CNF Satisfiability Algorithms and Properties

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# Why is it Worthwhile?

## The Questions:

- Why are some problems so difficult?
- Are there algorithms that will make them easier?

## Why Probability?

- Results and process tend to draw out intuition
  - Identify properties that may be exploited by a fast algorithm and properties that may prevent exploitation.
  - Identify reasons for the hardness of various problems - why **lots** of instances are hard.
- Can explain the good or bad behavior of an algorithm
- Afford comparison of incomparable classes of formulas
- De-randomization may yield fast algorithms

# Terms

A **variable** looks like this: $v_9$, takes a value from $\{0,1\}$

A **positive literal**: $v_9$, a **negative literal**: $\neg v_9$

A **clause** looks like this: $(\neg v_1 \vee \neg v_2 \vee v_5 \vee v_9)$

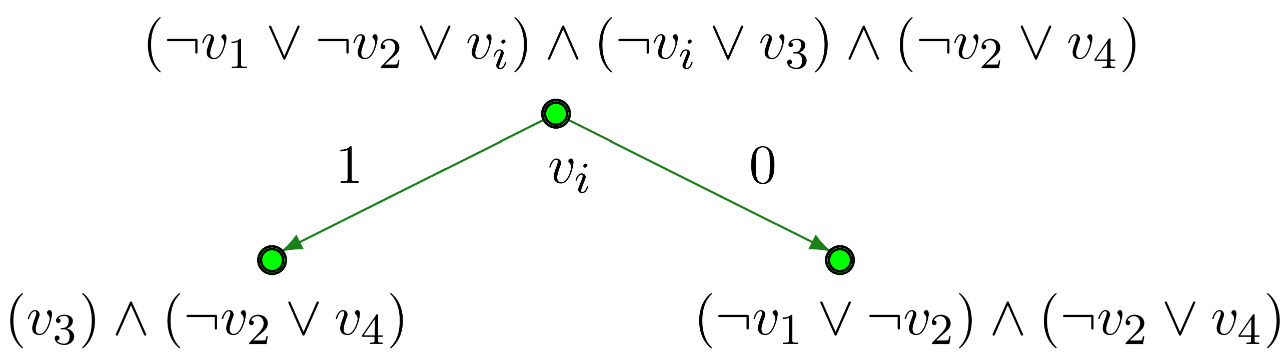
An **instance** of SAT looks like this:

$(v_1 \vee \neg v_2 \vee v_7) \wedge (\neg v_2 \vee v_6) \wedge (\neg v_2 \vee \neg v_4 \vee \neg v_5) \wedge (v_{10})...$

Clause **width**: # literals; **$k$-SAT**: fixed width $k$

An assignment of values satisfying $\mathcal{F}$ is a **model** for $\mathcal{F}$

An important **splitting** operation in solvers:



# Some problems

- Must assume an input distribution,
  often not reflecting reality
  (but sometimes we do not need to)
- Analysis can be difficult or impossible -
  algorithmic steps may significantly change distribution
  (known tools are limited)
- Can yield misleading results

# A Misleading Result

**Example:** A probabilistic model for random formulas

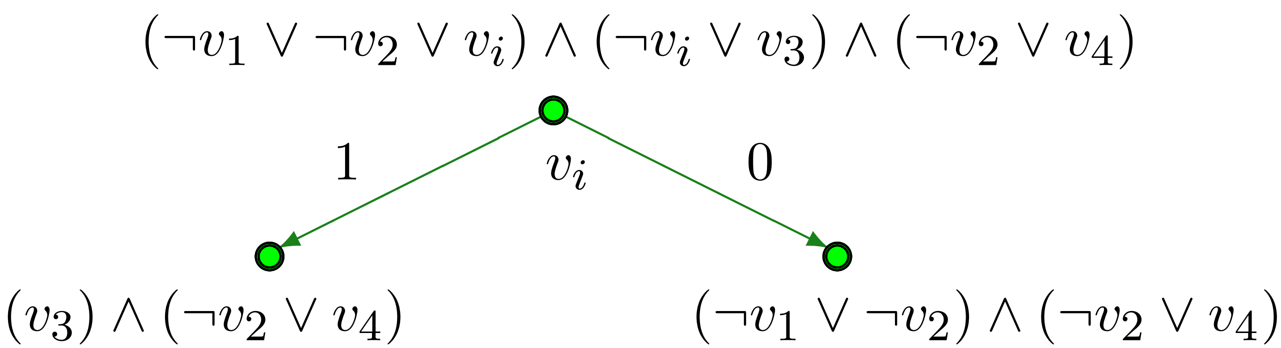
Given: set $L = \{v_1, \neg v_1, ..., v_n, \neg v_n\}$ of literals and $0 < p < 1$

Construct clause $c$: $l \in L$ independently in $c$ with probability $p$

Construct formula: $m$ independently constructed clauses

**Justification:** All formulas are equally likely if $p = 1/3$.

## Applied to splitting (DPLL):



# A Misleading Result

## Analysis sketch:

Given $m$ clauses, the average number of clauses removed when a value is assigned is $pm$

Let $T_i$ be the average number of clauses remaining on the $i^{th}$ iteration

Then $T_0 = m$ and $T_i = (1-p)T_{i-1}$

For what $i$ does $T_i = 1$? When $1 = m(1-p)^i$ or

$$\lg(m) = -i\lg(1-p)$$

$$i = \lg(m)/-\lg(1-p)$$

So, size of search space is $2^i \approx 2^{\lg(m)/p} = m^c$ if $p = 1/3$

**Conclusion:** *DPLL is fantastic, on the average!*

# A Misleading Result

## Problems with the analysis:

- The input model is funky!
  The probability that a random assignment satisfies a random formula is
  $$(1-(1-p)^{2n})^m \approx e^{-me^{-2pn}}$$
  which tends to $1$ if $\ln(m)/pn = o(1)$ and means the average width of a clause can be at least $\ln(m)$.
  With $p = 1/3$, this holds if $n > \ln(m)$.

- If average clause width is constant $\left(p = c/n\right)$ then the average search space size is
  $$2^{-\lg(m)/\lg(1-p)} \approx 2^{\lg(m)/(c/n)} = 2^{n\lg(m)/c} = m^{n/c}$$

**Exponential complexity!**

# Probabilistic Toolbox

## Linearity of expectation:

$$E\{\sum_i X_i\} = \sum_i E\{X_i\}, \quad X_i \text{ real valued r.v.s}$$

**First Moment Method:** show $Pr(P) \rightarrow 0$ as $n \rightarrow \infty$.

Let $X$ be a count of some entity

Suppose some property $P$ holds if and only if $X \geq 1$.

Then

$$Pr(X > 1) \leq E\{X\}$$

So, if $E\{X\} \rightarrow 0$ then $Pr(P) \rightarrow 0$, as $n \rightarrow \infty$.

# First Moment Method

## An Example:

Assume $\mathcal{F}$ is a random $k$-SAT formula, $m$ clauses, $n$ variables

Let $P$ be the property that $\mathcal{F}$ has a model

Let $X$ be the number of models for $\mathcal{F}$

Let $X_i = \begin{cases} 1 & \text{if the } i^{th} \text{ assignment is a model for } \mathcal{F} \\ 0 & \text{otherwise} \end{cases}$

$$Pr(X_i = 1) = (1 - 2^{-k})^m$$

$$E\{X\} = \sum_{i=1}^{2^n} Pr(X_i = 1) = 2^n(1 - 2^{-k})^m$$

$Pr(\mathcal{F}$ has a model$) \rightarrow 0$ if $\dfrac{m}{n} > \dfrac{1}{\lg(1 - 2^{-k})} \approx 2^k$

For $k = 3$ $\mathcal{F}$ has no model w.h.p. if $\dfrac{m}{n} > 5.19$

# Probabilistic Toolbox

## Flow Analysis:

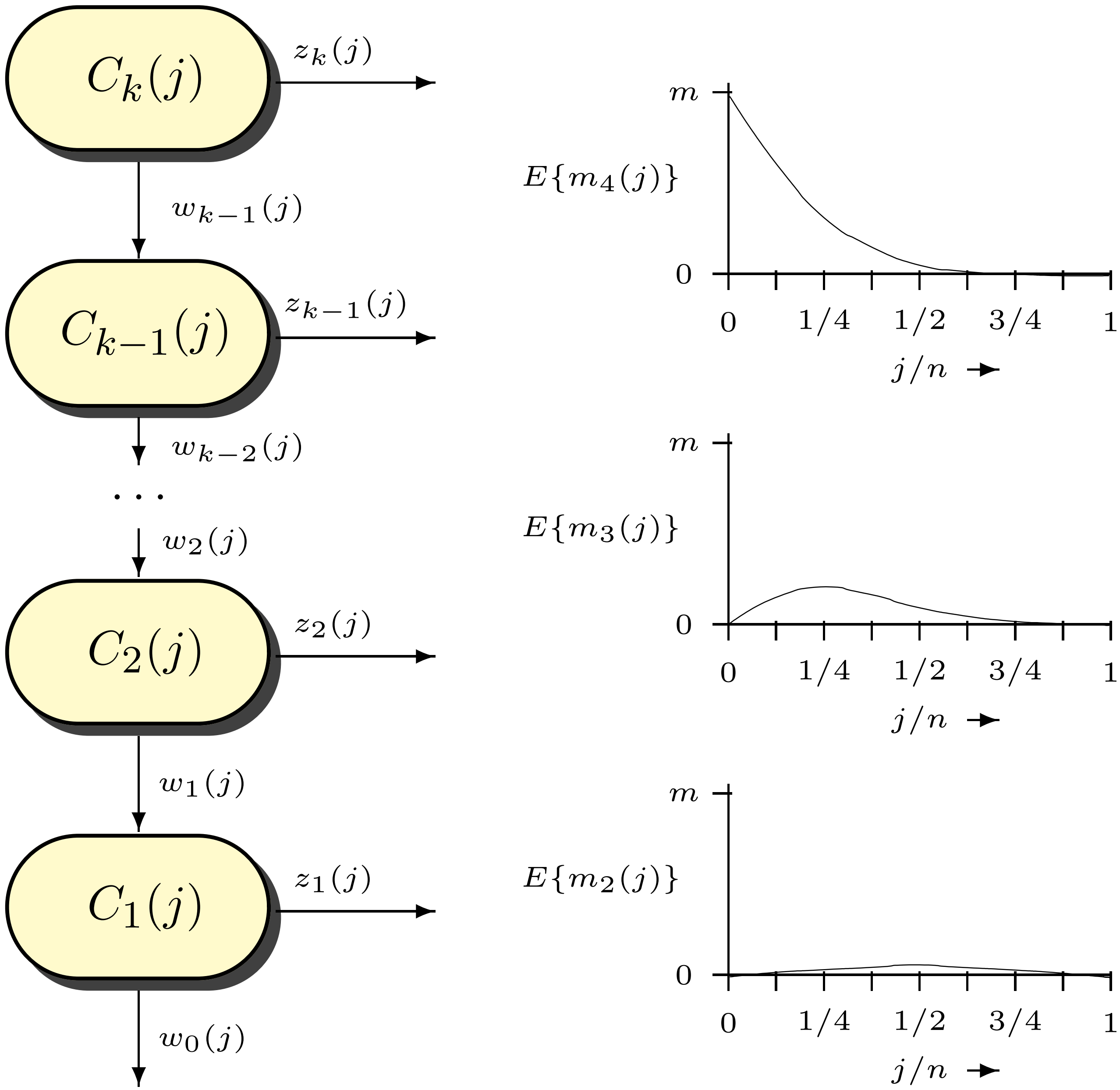
Consider straight-line (non-backtracking) variants of DPLL

Let $\mathcal{F}$ be a CNF Boolean expression
Set $M = \emptyset$
Repeat the following:
    Choose an unassigned literal $l$ in $\mathcal{F}$
    If $l$ is a positive literal, set $M = M \cup \{l\}$
    Set $\mathcal{F} = \{c \setminus \{\neg l\} : c \in \mathcal{F}, l \notin c\}$
    If $\mathcal{F}$ is satisfied then return $M$
    If some clause in $\mathcal{F}$ is falsified return "give up"

When does this <u>not</u> give up with probability tending to 1?

Answer depends on the way literals are chosen

# Flow Analysis



# Flow Analysis

## Example:

**Unit clause heuristic**: When there is a clause with one unassigned variable remaining, set the value of such a variable so as to satisfy its clause.

**Intuitively:** If the clause flow $w_1(j) < 1$ then any accumulation of unit clauses can be prevented and no clauses will ever be eliminated

**Analysis:**

Write difference equations describing flows:

$$m_i(j+1) = m_i(j) + w_i(j) - w_{i-1}(j) - z_i(j), \quad \forall\ 0 \le i \le k, 1 < j < n$$

Take expectations and rearrange:

$$E\{m_i(j+1) - m_i(j)\} = E\{w_i(j)\} - E\{w_{i-1}(j)\} - E\{z_i(j)\}$$

# Flow Analysis

Compute the expectations:

$$E\{z_i(j)\} = E\{E\{z_i(j)|m_i(j)\}\} = E\left\{\frac{i \cdot m_i(j)}{2(n-j)}\right\} = \frac{i \cdot E\{m_i(j)\}}{2(n-j)}$$

$$E\{w_i(j)\} = E\{E\{...\}\} = E\left\{\frac{(i+1)m_{i+1}(j)}{2(n-j)}\right\} = \frac{(i+1)E\{m_{i+1}(j)\}}{2(n-j)}$$

Substitute into difference equations:

$$E\{m_i(j+1) - m_i(j)\} = \frac{(i+1)E\{m_{i+1}(j)\}}{2(n-j)} - \frac{i \cdot E\{m_i(j)\}}{n-j}, \qquad i < k$$

$$E\{m_k(j+1) - m_k(j)\} = -\frac{k \cdot E\{m_k(j)\}}{n-j}$$

Switch to differential equations (use $x$ for $j/n$):

$$\frac{d\bar{m}_i(x)}{dx} = \frac{(i+1)\bar{m}_{i+1}(x)}{2n(1-x)} - \frac{i \cdot \bar{m}_i(x)}{n-j}; \quad \bar{m}_k(0) = m/n,\ \bar{m}_i(0) = 0 \text{ for } i < k$$

Solve:

$$\bar{m}_i(x) = \frac{1}{2^{k-i}}\left(\frac{m}{n}\right)\binom{k}{i}(1-x)^i x^{k-i}$$

Translation:

$$E\{m_i(j)\} = \frac{m}{2^{k-i}}\binom{k}{i}\left(1-\frac{j}{n}\right)^i\left(\frac{j}{n}\right)^{k-i}$$

# Flow Analysis

The important flow:

$$E\{w_1(j)\} = \frac{E\{m_2(j)\}}{(n-j)} = \frac{1}{2^{k-2}} \binom{k}{2} \left(1 - \frac{j}{n}\right)^2 \left(\frac{j}{n}\right)^{k-2} m$$

Find location of maximum (set derivative = 0):

$$j^* = \left(\frac{k-2}{k-1}\right) n$$

So,

$$E\{w_1(j^*)\} < 1 \quad \text{when} \quad \frac{m}{n} < \frac{2^{k-1}}{k} \left(\frac{k-1}{k-2}\right)^{k-1}$$

for $k = 3$ this is $\frac{m}{n} < \frac{8}{3}$

Conclusion: unit clause heuristic succeeds with probability bounded by a constant when $\frac{m}{n} < \frac{8}{3}$.

# Flow Analysis

## What makes this work?

The clausal distribution is the same, up to parameters $m$ and $n$, at each level (algorithm is myopic - no information is revealed)

There is pretty much never a sudden spurious flow

$Pr(||C_i(j+1)| - |C_i(j)|| > n^{0.2}) = o(n^{-3})$

The average flow change is pretty smooth

$E\{|C_i(j+1)| - |C_i(j)|\} = f_i(j/n, |C_0(j)|/n, ..., |C_k(j)|/n) + o(1),$

$f_i$ is continuous and

$$|f_i(u_1, ..., u_{k+2}) - f_i(v_1, ..., v_{k+2})| \leq L \sum_{1 \leq i \leq j} |u_i - v_i|$$

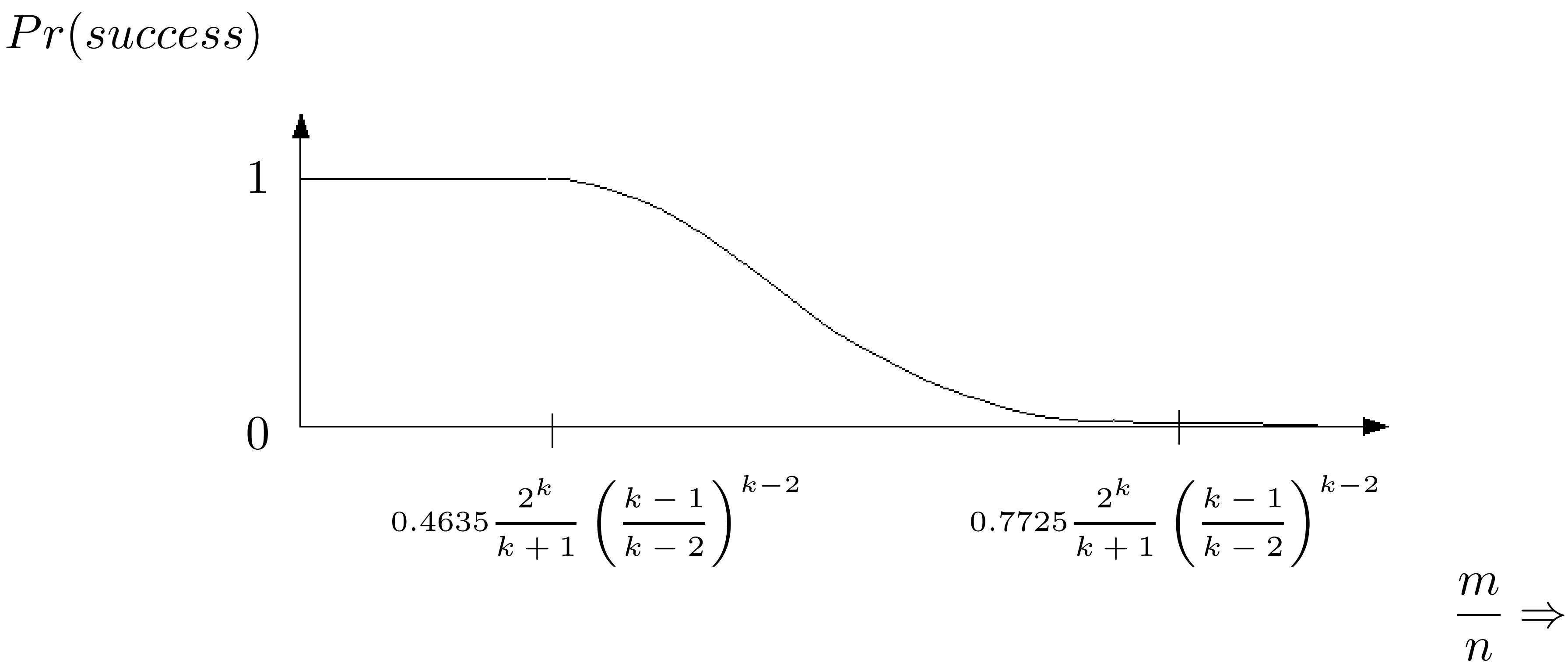
# Flow Analysis

| | good when $m/n <$ | |
|---|---|---|
| **literal selection** | $k$**-SAT** | **3-SAT** |
| Choose from unit clause, otherwise randomly | $2^k/k$ | 2.66 |
| Choose var with maximum difference between occurrences of positive and negative lits. Set value to maximize satisfied clauses | $c \cdot 2^k/k$ | 3.003 |
| Choose randomly from a clause with fewest non-falsified literals | $1.125 \cdot 2^k/k$ | 3.09 |
| Best possible myopic algorithm | $c \cdot 2^k/k$ | 3.26 |

## literal selection, non-myopic algorithms

| | | |
|---|---|---|
| Greedy algorithm: maximize number of clauses satisfied, eliminate unit clauses when they exist | $c \cdot 2^k/k?$ | 3.52 |
| Maximize the expected number of models of the reduced instance | $c \cdot 2^k/k?$ | $> 3.6?$ |

# Flow Analysis




Typical probability curve of these algorithms

Is $\frac{2^k}{k}$ the crossover to unsatisfiability or is it $2^k$?

# Probabilistic Toolbox

**Second Moment Method:** show $Pr(P) \to 1$ as $n \to \infty$

Let $P$ be some property of a formula

A **witness** for $P$: a structure whose presence in $\mathcal{F}$ implies $P$

Let $W = \{w : w$ is a witness for $P$ in $\mathcal{F}\}$

Let $X_i = \begin{cases} 1 & \text{if structure } s_i \in W \\ 0 & \text{otherwise} \end{cases}$

Let $X = \sum_i X_i$ and $E\{X_i\} = q$, then $E\{X\} \triangleq \mu = q|W|$

Suppose $var(X) \triangleq \sigma^2 = o(\mu^2)$. Then, since

$$Pr(X = 0) \leq \frac{\sigma^2}{\mu^2}$$

we get

$$Pr(P) \to 1 \text{ as } n \to \infty$$

# Second Moment Method

**To show $\sigma^2 = o(\mu^2)$:**

Start with one witness $w$ chosen arbitrarily

Let $A_w$ be all witnesses sharing at least one clause with $w$

Let $D_w$ be all witnesses sharing no clause with $w$. Then

$$\sigma^2 = \mu(1-q) + \mu \left( \sum_{z \in A_w} (Pr(z|w) - q) + \sum_{z \in D_w} (Pr(z|w) - q) \right)$$

Need $|A_w| \ll |D_w|$ or "little" overlap among witnesses since

$Pr(z|w) = Pr(z) = q$ if $z \in D_w$ so $\sum_{z \in D_w} (Pr(z|w) - q) = 0$.

If $\mu \to \infty$ and $\displaystyle\sum_{z \in A_w} Pr(z|W) \to o(\mu)$ as $n \to \infty$ then

$Pr(P) \to 1$ as $n \to \infty$

# Where is the Crossover?

Cannot apply the second moment method directly to $k$-SAT
- variance of the number of models is too high
  the reason: the asymmetry of $k$-SAT

Let $z$ and $w$ be assignments agreeing in $\alpha n$ variables

$$Pr(z \text{ satisfies } \mathcal{F} \mid w \text{ satisfies } \mathcal{F}) = \left(1 - \frac{1-\alpha^k}{2^k-1}\right)^m$$

Variance gets too big, maximum occurs at $\alpha \neq 1/2$

$$\sum_{z \in A_w} Pr(z|w) = \sum_{0<\alpha\leq 1} \binom{n}{\alpha n} \left(1 - \frac{1-\alpha^k}{2^k-1}\right)^m$$

# Where is the Crossover?

Analyze a different problem: Not All Equal $k$-SAT

A NAE model: one for which every clause has at least one true and at least one false literal

$$Pr(\exists \text{ a model for } \mathcal{F}) > Pr(\exists \text{ a NAE-model for } \mathcal{F})$$

Let $X$ = number of models, $X_{NAE}$ = number of NAE models

$$Pr(X_{NAE} = 0) > Pr(X = 0)$$

$$Pr(z \text{ NAE-satisfies } \mathcal{F} \mid w \text{ NAE-satisfies } \mathcal{F}) = \left(1 - \frac{1 - \alpha^k - (1-\alpha)^k}{2^{k-1} - 1}\right)^m$$

Variance is low, maximum term occurs at $\alpha = 1/2$.

$$\sum_{0<\alpha\leq 1} \binom{n}{\alpha n} \left(1 - \frac{1 - \alpha^k - (1-\alpha)^k}{2^{k-1} - 1}\right)^m \approx \frac{2^n(1 - 2^{1-k})^m}{\sqrt{n}} \approx o(\mu)$$

$$Pr(X_{\mathrm{NAE}} = 0) < \frac{1}{\mu_{\mathrm{NAE}}} \approx \frac{1}{2^n(1 - 2^{k-1})^m} \quad \to 0 \text{ if } \frac{m}{n} > \frac{1}{\lg(1 - 2^{k-1})} \approx 2^{k-1}$$

# What About Easy Classes?

Examples:

Horn, Hidden Horn, 2-SAT, Extended Horn, q-Horn, CC-balanced, SLUR, Matched, Linear Autarkies

Horn:

every clause has at most one positive literal
solved in linear time by unit clause algorithm

Probabilistic analysis of polytime solvable classes can reveal:

- What critically distinguishes an easy class from more difficult classes
- Whether one class is much larger than another incomparable class in a probabilistic sense

# Polynomial Time Solvable Classes

## Example: q-Horn

variables

$$
\begin{array}{r}
\text{Horn} \\ \\ \\ \text{clauses} \\ \\ \\ \\ \text{non-positive}
\end{array}
\left\{
\begin{array}{rrrrr|rrrrrr}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
-1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{array}
\right\}
\begin{array}{l}
\text{Zero} \\ \\ \\ \\ \\ \\ \\ \text{2-SAT}
\end{array}
$$

If the satisfiability index of a given formula is no greater than $1$, then the formula is q-Horn.

The class of formulas with a satisfiability index greater than $1 + n^{-\epsilon}$, for any $\epsilon$, is NP-complete.

# Polynomial Time Solvable Classes

## Vulnerability of q-Horn to particular cycles

$$
\begin{tikzcd}
\ldots\ (u_2 \vee \neg u_1 \vee \ldots) \arrow[r, dash, "u_1"] & (u_1 \vee \neg v_0 \vee \neg u_{3p} \vee \ldots) \arrow[r, dash, "u_{3p}"] \arrow[d, dash, "v_0"] & (u_{3p} \vee \neg u_{3p-1} \vee \ldots)\ \ldots \\
\ldots\ (\neg u_{p-1} \vee u_p \vee \ldots) \arrow[r, dash, "u_p"'] & (\neg u_p \vee \neg v_0 \vee u_{p+1} \vee \ldots) \arrow[r, dash, "u_{p+1}"'] & (\neg u_{p+1} \vee u_{p+2} \vee \ldots)\ \ldots
\end{tikzcd}
$$

| | $Pr$(random formula is of specified class) | | |
|---|---|---|---|
| **class** | **as** $n \to \infty$ | $k$**-SAT** | **3-SAT** |
| Horn | 0 | $m > \epsilon$ | $m > \epsilon$ |
| Hidden Horn | 0 | $m/n > 1/(k - \lg(k+1))$ | $m/n > 1$ |
| q-Horn | 0 | $m/n > 4/(k^2 - k)$ | $m/n > 0.66$ |
| SLUR | 0 | $m/n > 4/(k^2 - k)$ | $m/n > 0.66$ |
| Matched | 1 | $m/n < c_k,\ c_k \to 1$ | $m/n < 0.64$ |
| No Cycles | 1 | $m/n < 1.36/(k^2 - k)$ | $m/n < 0.226$ |

# Algorithms for Unsatisfiability

## Resolution performs badly on random unsatisfiable formulas

$$Pr(\text{random } k\text{-SAT formula is unsatisfiable}) \to 1 \quad \text{if} \quad \frac{m}{n} > 2^k$$

$$Pr(\text{resolution does well}) \to 1 \quad \text{only if} \quad \frac{m}{n} > \left(\frac{n}{\lg(n)}\right)^{k-2}$$

## Are there better alternatives?

Must avoid getting stuck on "sparse" nature of formulas

## One possibility: Hitting Set

Focus attention on clauses which are all positive or negative

Let $n^+ =$ min number of $1$ valued variables to satisfy positives

Let $n^- =$ min number of $0$ valued variables to satisfy negatives

If $n^+ + n^- > n$ then some variable must be set to $1$ AND $0$

That is impossible, conclude the formula is unsatisfiable

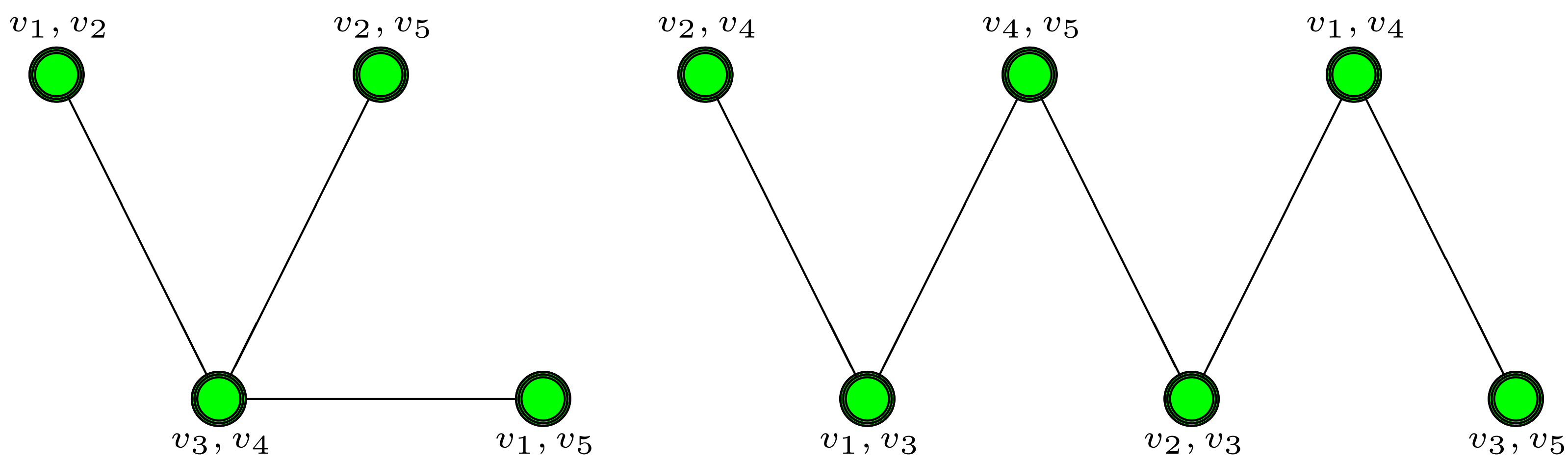
# Hitting Set

Construct graphs $G^+, G^-$

Vertices are labeled as pairs of variables

Edge $\langle a, b \rangle \Leftrightarrow$ some clause contains all variables labeling $a, b$

$$(v_1 \vee v_2 \vee v_3 \vee v_4) \wedge (v_2 \vee v_3 \vee v_4 \vee v_5) \wedge (v_1 \vee v_3 \vee v_4 \vee v_5)$$



# Hitting Set

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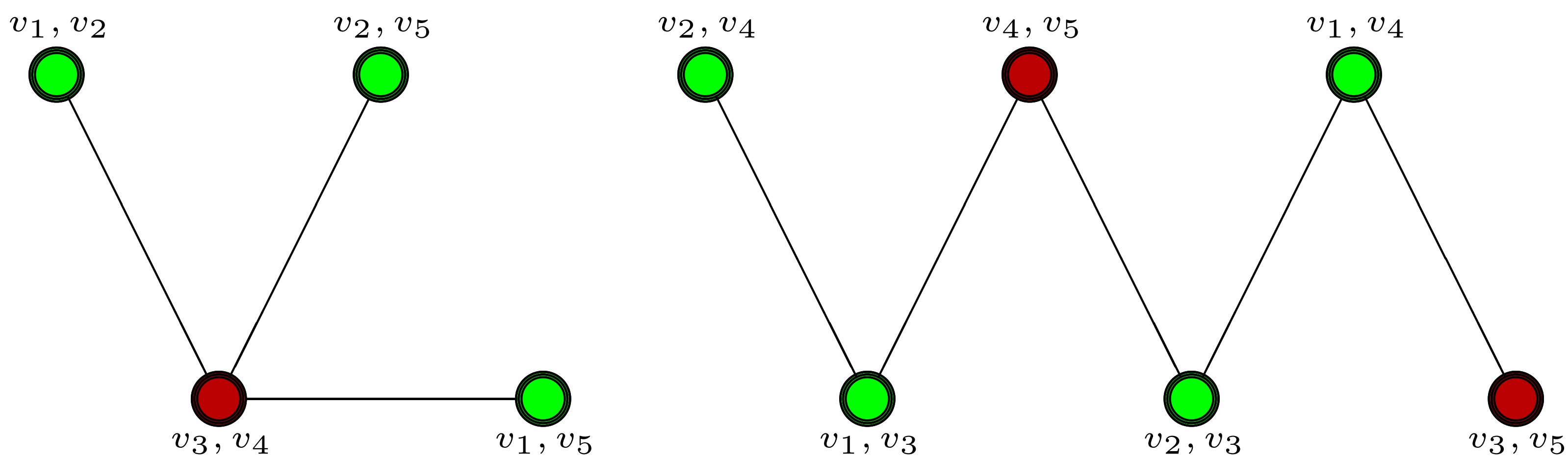
$$(v_1 \vee v_2 \vee v_3 \vee v_4) \wedge (v_2 \vee v_3 \vee v_4 \vee v_5) \wedge (v_1 \vee v_3 \vee v_4 \vee v_5)$$




For $k = 4$, if a model exists, one graph has $|IS| > \frac{n^2}{8}$

# Hitting Set

Construct matrices $M^+, M^-$

Columns and rows are indexed on vertices

$$M_{i,j} = \begin{cases} -\frac{1-p}{p} & \text{if there is an edge between vertices } i \text{ and } j \\ 1 & \text{otherwise} \end{cases}$$

where $p$ represents a probability that can be adjusted

Exploit relationship between maximum eigenvalue and maximum IS

$$\alpha(G^+) < \lambda_1(M^+) \quad \text{and} \quad \alpha(G^-) < \lambda_1(M^-)$$

For purposes of proving a bound

If $p = \dfrac{\ln^7(n')}{n'}$ then $\max_i\{|\lambda_i(M)|\} = \dfrac{2n'}{\ln^{3.5}(n')}(1 + o(1))$ w.h.p.

This leads to $Pr(HS\ does\ well) \to 1$ if $\dfrac{m}{n} > n^{(k-2)/2}$

Recall $Pr(resolution\ does\ well) \to 1$ only if $\dfrac{m}{n} > n^{k-2}$.

# A De-randomized Algorithm for MAXSAT

## MAX $k$-SAT

Given: A CNF formula $\mathcal{F}$ with $k$ literals per clause

Find: An assignment to the variables of $\mathcal{F}$ that satisfies a maximum number of its clauses.

Suppose $\mathcal{F}$ has variables $v_1, v_2, \ldots, v_n$. Define indicators

$$A^i_{t_1,\ldots,t_j} = \begin{cases} 1 & \text{if clause } i \text{ satisfied given } v_1 = t_1, ..., v_j = t_j \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that $A^i_{t_1,\ldots,t_j} = 1$ for random $t_{j+1}, ..., t_n$?

for example: $Pr((\neg v_1 \vee v_3 \vee \neg v_5) = 1 \mid t_1 = 1, t_2 = 0) = Pr(A^i_{1,0}) = 3/4$

# A De-randomized Algorithm for MAXSAT

Let $N$ be the number of satisfied clauses in $\mathcal{F}$. Then

$$E\{N\} = \sum_{i=1}^{n} Pr(A^i) = (1 - 2^{-k})m$$

and

$$\begin{aligned} Pr(A^i_{t_1,\ldots,t_{j-1}}) &= (Pr(A^i_{t_1,\ldots,t_{j-1},1}) + Pr(A^i_{t_1,\ldots,t_{j-1},0}))/2 \\ &\leq \text{max } \{Pr(A^i_{t_1,\ldots,t_j})\} \end{aligned}$$

so keep choosing a value $t_j$, for $j = 1, 2, \ldots$ that maximizes $Pr(A^i_{t_1,\ldots,t_j})$ to get

$$E\{N\} = (1 - 2^{-k})m = \sum_{i=1}^{n} Pr(A^i) \leq \sum_{i=1}^{n} \text{max } \{Pr(A^i_{t_1})\} \ldots$$

$$\leq \sum_{i=1}^{n} \text{max } \{Pr(A^i_{t_1,\ldots,t_n})\} = \# \text{ clauses satisfied given } t_1, \ldots, t_n$$

# A De-randomized Algorithm for MAXSAT

MAX $k$-SAT approximation algorithm uses this:

```
bool chooseValue (Variable *var) {
   double sum_pos=0.0, sum_neg=0.0, prob=0.5;
   for (int sz=1 ; sz <= k ; sz++) {
      sum_pos += var->no_clauses_as_pos_lit[sz]*prob;
      sum_neg += var->no_clauses_as_neg_lit[sz]*prob;
      prob *= 0.5;
   }
   return (sum_pos >= sum_neg) ? true : false;
}
```

and will always find an assignment that satisfies at least $(1 - 2^{-k})m$ clauses - for any input formula!