Abstract

Conditioning is an important task for designing intelligent systems in artificial intelligence. This paper addresses an issue related to the possibilistic counterparts of Jeffrey's rule of conditioning. More precisely, it addresses the existence and unicity of solutions computed using the possibilistic counterparts of the so-called $kinematics$ properties underlying Jeffrey's rule of conditioning. We first point out that like the probabilistic framework, in the quantitative possibilistic setting, there exists a unique solution for revising a possibility distribution given uncertainty bearing on a set of exhaustive and mutually exclusive events. However, in the qualitative possibilistic framework, the situation is different. In particular, the application of the kinematics principle does not guarantee the existence of a solution. We provide precise conditions where the unicity of the revised possibility distribution exists.

On analysis of the unicity of Jeffrey's rule of conditioning in a possibilistic framework

Salem Benferhat, Karim Tabia, Karima Sedki
CRIL CNRS UMR 8188 - Université d'Artois
{benferhat,tabia,sedki}@cril.univ-artois.fr

Introduction

Probability theory is among the main framework for representing and reasoning with uncertain information. Beliefs can be represented by means of probability distributions and they can be revised and updated in the presence of new information. When the considered new piece of information is fully certain (evidence), then one can update the current beliefs by means of conditioning. In case where the piece of information in hand is uncertain (for instance, represented by a probability distribution), then the current beliefs are $revised$ according to the new information. In probability theory, there are two main and equivalent methods for achieving this task. The first one is Jeffrey's rule (Jeffrey 1965) for revising probability distribution which is based on probability kinematics principle (Domotor 1980) whose objective is minimizing belief change. In this method, beliefs are represented as a probability distribution. The second one is Pearl's method of virtual evidence (Pearl 1988) proposed in the context of probabilistic graphical models. In (Chan & Darwiche 2005)(Darwiche 2009), the authors show that the two methods are equivalent and aim at minimizing belief change and only differ in the way uncertain evidence is specified. Other approaches have also been proposed to encode and manage uncertain evidence such as Vomlel's soft evidence approach (Vomlel 2004).

Possibility theory is an elegant alternative framework suitable for handling uncertain, imprecise and incomplete knowledge. In possibility theory, there are two different ways to define conditioning depending on how possibility degrees are interpreted (either we fully use the numerical unit interval [0, 1] or we only use a relative total preordering induced by the unit interval [0, 1]. This distinction leads to two different frameworks called quantitative (or product-based) framework and qualitative (or min-based) one (see (Dubois & Prade 1988) for more details)). The possibilistic counterpart of Jeffrey's rule was directly used in (Dubois & Prade 1997)(Dubois & Prade 1993) and different ways for defining conditioning with uncertain evidence have been proposed. However, neither a real reference to probability kinematics nor an analysis of the unicity of the solution have been performed. This paper investigates the existence and unicity of solutions computed using Jeffrey's rule in both the quantitative and qualitative possibilistic frameworks. We show that in the quantitative setting, we get similar results as in the probabilistic framework. However, in the qualitative setting, the existence and the unicity of the solution based on the possibilistic conditioning (under uncertain inputs) satisfying the $kinematics$ requirements is no longer guaranteed. In fact, kinematics conditions underlying Jeffrey's rule of conditioning is too strong in the qualitative framework. In order to satisfy these conditions, the inputs should reinforce the possibility degree of each interpretation of the language.

The rest of this paper is organized as follows: Section 2 briefly presents probability kinematics and Jeffrey's rule in the probabilistic framework. Section 3 studies the possibilistic counterpart of Jeffrey's rule and the existence and unicity of its solution in the quantitative possibilistic frameworks while section 4 analyzes the unicity of the solution in the qualitative setting. Section 5 concludes this paper.

Probability kinematics and Jeffrey's rule

Kinematics is a branch of physics describing the motion of systems without considering the facts causing the motion. Kinematics is usually considered as the contrary of dynamics which examines the causes and the consequences. According to Richard Jeffrey (Jeffrey 1965), probability kinematics constitutes a set of rules for changing a $prior$ probability distribution into a new $updated$ or $posterior$ distribution given a set of $imposed\ constraints$. It is important to note that in Jeffrey's view, uncertain evidence is specified as a $constraint$ in the sense that the new information must be completely accepted. This interpretation of viewing uncertain inputs as constraints that should be accepted may appear too strong in some applications. However, in other applications such handling interventions in graphical models (Pearl 2000) (Benferhat & Smaoui 2007), viewing uncertain inputs as strong constraints fully makes sense. Indeed, in graphical models an intervention on a variable A is handled as a set of strong constraints where beliefs on parents of A remain unchanged. Clearly, this is in the spirit of Jeffrey's rule. The reader can refer to (Dubois & Prade 1997) where the inputs are not required to be accepted.

Jeffrey's conditionalization rule (Jeffrey 1965) allows revising a probability distribution p into p' given uncertainty bearing on a set of mutually exclusive and exhaustive events λ_i. This method involves:

1. **A way for specifying uncertain evidence:** The uncertainty is of the form (λ_i, α_i) with α_i=$p'(\lambda_i)$ meaning that after the revision operation, the probability of each event λ_i must be equal to α_i (namely, $p'(\lambda_i)$=α_i). Hence, this way for specifying uncertainty relative to uncertain evidence expresses the fact that the uncertain evidence is seen as a constraint or an effect once the new evidence is accepted (Chan & Darwiche 2005).
2. **A method for computing the revised probability distributions:** Jeffrey's method assumes that although there is uncertainty about events λ_i in the old and new probability distributions, the conditional probability of any event $\phi \subseteq \Omega$[1] given any uncertain event λ_i remains the same in the original and the revised distributions. Namely,

$$\forall \lambda_i \in \Omega, \forall \phi \subseteq \Omega, P(\phi|\lambda_i) = P'(\phi|\lambda_i). \quad (1)$$

The underlying interpretation of revision implied by constraint of Equation 1 is that revised probability distribution p' must not change conditional probability degrees of any event ϕ given uncertain events λ_i.
In the probabilistic framework, applying Bayes rule then marginalization allows to revise the possibility degree of any event $\phi \subseteq \Omega$ in the following way:

$$P'(\phi) = \sum_{\lambda_i} P'(\lambda_i) * \frac{P(\phi, \lambda_i)}{P(\lambda_i)}. \quad (2)$$

Revised probability distribution p' obtained using Jeffrey's rule is unique (see the proof of the unicity in (Chan & Darwiche 2005)).

Example 1 *Let us give a running example which will be used to illustrate our results in the rest of this paper.*
Let us assume that the current beliefs about smoking and lung cancer decease are specified using two variables S (for smoking which can be either $True$ or $False$) and C (for lung cancer and also takes $True$ or $False$ value to express that the individual has lung cancer or not). Assume that a doctor expresses his beliefs about this problem by the joint probability distribution of Table 1: Note that from the joint

S (Smoking)	C (Lung Cancer)	p(S,C)
True	True	0,1
True	False	0,32
False	True	0,06
False	False	0,52

Table 1: Original probability distribution p encoding the initial beliefs

distribution p of Table 1, the marginal distribution relative to variable S is $p(S$=$True)$=.42 and $p(S$=$False)$=.58. Assume now that recent and $very$ $reliable$ surveys conclude that there are only 34% of individuals who are smokers (and 66% who do not smoke). The doctor can revise his initial beliefs[2] *in order to comply with the new survey results. Namely, the new probability distribution p' should lead to $p'(S$=$True)$=.34 and $p'(S$=$False)$=.66. This can be done using Jeffrey's rule given that uncertainty is bearing on an exhaustive and mutually exclusive set of events (namely, S=$True$ and S=$False$). The obtained probability distribution p' is given in Table 2. One can easily check*

S (Smoking)	C (Lung Cancer)	p'(S,C)
True	True	0,08
True	False	0,26
False	True	0,07
False	False	0,59

Table 2: Probability distribution p' encoding the revised beliefs

that $p'(S$=$True)$=.34 and $p'(S$=$False)$=.66 and $\forall$ $\phi \subseteq \Omega$, $P(\phi|S)$=$P'(\phi|S)$. In this situation, Jeffrey's rule preserves the conditional probability of having (or not) lung cancer given that the individual is smoker (or not). The intuitive interpretation is that prior knowledge on the proportion of smokers changes while the probability of having (or not) lung cancer given that the person smokes (or not) remains unchanged.

Possibilistic counterparts of Jeffrey's rule

Brief refresher on possibility theory

Possibility theory was introduced by Zadeh (Zadeh 1978) and developed by several researchers, eg. Dubois and Prade (Dubois & Prade 1988). It is an uncertainty theory based on a pair of dual measures in order to evaluate knowledge/ignorance relative to event in hand. The concept of possibility distribution π is one of the important building blocks of possibility theory: It is a mapping from the universe of discourse Ω to the unit scale $[0, 1]$ which can be either quantitative or qualitative. In both these settings, a possibility degree $\pi(w_i)$ expresses to what extent it is consistent that w_i can be the actual state of the world. In particular, $\pi(w_i)$=1 means that w_i is totally possible and $\pi(w_i)$=0 denotes an impossible event. The relation $\pi(w_i)$>$\pi(w_j)$ means that w_i is more possible than w_j. A possibility distribution π is said to be normalized if $max_{w_i \in \Omega}(\pi(w_i))$=1. It is said to be sub-normalized otherwise.
The second important concept in possibility theory is the one of possibility measure denoted $\Pi(\phi)$ and computing the possibility degree relative to an event $\phi \subseteq \Omega$. It evaluates to what extent ϕ is consistent with the current knowledge encoded by possibility distribution π on Ω. It is defined as follows:

$$\Pi(\phi) = max_{w_i \in \phi}(\pi(w_i)). \quad (3)$$

[1] Ω represents the universe of discourse, all the possible states of the world.

[2] In this paper, we do not go through a discussion regarding the differences between factual situations and knowledge. In this example (and more generally in Jeffrey's framework), the input more concerns a piece of knowledge to be taken into account, instead of an information regarding some particular situation. In this paper, we use the term beliefs for inputs that concern both factual and knowledge. See (Dubois 2008) for more details.

The term $\Pi(\phi)$ denotes the possibility degree relative to having one of the events involved in ϕ as the actual state of the world.
The necessity measure is the dual of possibility measure and evaluates the certainty implied by the current knowledge of the world. Namely, $N(\phi)$=1-$\Pi(\overline{\phi})$ where $\overline{\phi}$ denotes the complementary of ϕ.
Given a possibility distribution π on Ω, marginal distributions π_X relative to a subset of variables X ($X \subseteq V$) are computed using the max operator as follows:

$$\pi_X(x) = max_{w_i \in \Omega}(\pi(w_i) : w_i[X] = x), \tag{4}$$

where term $w_i[X] = x$ denotes the fact that x is the instantiation of X in w_i.
According to the interpretation underlying the possibilistic scale [0,1], there are two variants of possibility theory:

- **Qualitative (or min-based) possibility theory**: In this case, the possibility distribution is a mapping from the universe of discourse Ω to an "ordinal" scale where only the "ordering" of values is important.
- **Quantitative (or product-based) possibility theory**: In this case, the possibilistic scale [0,1] is numerical and possibility degrees are like numeric values that can be manipulated by arithmetic operators.

These two frameworks mainly differ in the way conditioning is defined. More details on the possible interpretations of possibility degrees can be found in (Dubois & Prade 1988).

Jeffrey's rule of conditioning in the possibilistic framework

In the framework of possibility theory, generic knowledge can be encoded by specifying a possibility distribution. Given the initial beliefs encoded by a possibility distribution π and uncertainty bearing on an exhaustive and mutually exclusive set of events λ_i in the form of (λ_i, α_i) such that $\pi'(\lambda_i)$=α_i, then the revised possibility distribution π' according to Jeffrey's rule must satisfy the following conditions:

i) $\forall \lambda_i, \pi'(\lambda_i) = \alpha_i$.

ii) $\forall \lambda_i \in \Omega, \forall \phi \subseteq \Omega, \Pi'(\phi|\lambda_i) = \Pi(\phi|\lambda_i)$.

As in the probabilistic framework, revising a possibility distribution π into π' according to the possibilistic counterpart of Jeffrey's rule must comply with the principle stating that uncertainty about events λ_i must not alter the conditional possibility degree of any event $\phi \subseteq \Omega$ given any uncertain event λ_i.
The possibilistic counterpart of Jeffrey's rule was directly used in (Dubois & Prade 1997) without neither a real reference to probability kinematics nor an analysis of the existence/unicity of the solution. More recently, authors in (Benferhat *et al.* 2009) investigate possibilistic belief revision based on the possibilistic counterparts of Jeffrey's and show that this rule can successfully recover most of the beliefs revision kinds such as the natural belief revision (Boutilier 1993), drastic belief revision (Papini 2001), reinforcement (Konieczny & Perez 2008), etc. In the following, we will study the existence and unicity of the revised possibility distribution computed using the counterparts of Jeffrey's rule in the possibilistic framework.

Product-based possibilistic counterpart of Jeffrey's rule

The product-based possibilistic framework has several similarities with the probabilistic one. This setting uses the $product$ operator and conditioning is defined as follows (we assume that $\Pi(\phi)$>0):

$$\pi_p(w_i|\phi) = \begin{cases} \frac{\pi(w_i)}{\Pi(\phi)} & \text{if } w_i \in \phi; \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

where $w_i \in \Omega$ stands for an elementary world and $\phi \subseteq \Omega$ is an arbitrary event.
Revision based on the possibilistic counterpart of Jeffrey's rule in the product-based framework can be formalized as follows:

Definition 1 Let π be a possibility distribution and (λ_1, α_1),..,(λ_n,α_n) be a set of exhaustive and mutually exclusive events where uncertainty is of the form $\pi'(\lambda_i)$=α_i for i=1..n. The revised possibility degree of any arbitrary event $\phi \subseteq \Omega$ is computed using the following formula:
$\forall \phi \subseteq \Omega$, $\Pi'(\phi)$=$max_{\lambda_i}(\alpha_i * \frac{\Pi(\phi,\lambda_i)}{\pi(\lambda_i)})$.

Note that Definition 1 has been proposed in several frameworks (for more details see (Smets 1993)).

Example 2 *Let us use the same example as in the probabilistic framework. Assume that the doctor expresses his beliefs using the joint possibility distribution of Table3.*
In the distribution π of Table 3, we have $\Pi(S$=$True)$=.8

S (Smoking)	C (Lung Cancer)	π(S,C)
True	True	0,4
True	False	0,8
False	True	0,2
False	False	1

Smoking	π(S)
True	0,8
False	1

π(C|S)
0,5
1,0
0,2
1,0

Table 3: Possibility distribution π encoding the initial beliefs

and $\Pi(S$=$False)$=1 meaning that the statement "an individual is not a smoker" *is totally possible while the statement "the individual is smoker" is plausible with only a degree of .8. Assume now that recent and reliable surveys conclude that there are only half less individuals who are smokers than in the previous surveys. The doctor then revises his initial beliefs in order to comply with the new surveys conclusions. Namely, the new possibility distribution π' should lead to $\Pi'(S$=$True)$=.4 and $\Pi'(S$=$False)$=1. This can be done using Jeffrey's rule of Definition 1. The obtained possibility distribution π' is given in Table 4. One can easily check that $\Pi'(S$=$True)$=.4 and $\Pi'(S$=$False)$=1 and $\forall \phi \subseteq \Omega$, $\Pi(\phi|S)$=$\Pi'(\phi|S)$. In this example, Jeffrey's rule preserved the conditional possibility degree of having (or not) lung cancer given that the individual is smoker (or not).*

S (Smoking)	C (Lung Cancer)	π'(S,C)
True	True	0,2
True	False	0,4
False	True	0,2
False	False	1,0

Smoking	π'(S)
True	0,4
False	1

π'(C|S)
0,5
1,0
0,2
1,0

Table 4: Revised possibility distribution π' encoding the revised beliefs

Note that the revised possibility distribution π' computed using Definition 1 is unique which is formalized in the following proposition:

Proposition 1 Let π be a possibility distribution and (λ_1, α_1),..,(λ_n, α_n) be a set of uncertain, exhaustive and mutually exclusive events where uncertainty is of the form $\pi'(\lambda_i)$=α_i for i=1..n. Then, there exists a unique possibility distribution π' (the one computed using equation of Definition 1) that satisfies conditions i) and ii).

Proof of Proposition 1

The proof is very similar to the one provided by (Chan & Darwiche 2005) in the probability theory framework. Let us prove Proposition 1 by first proving that the possibility distribution π' computed using equation of Definition 1 satisfies conditions i) and ii) then prove that a possibility distribution satisfying conditions i) and ii) is necessarily computed using equation of Definition 1.

- Let (λ_1,α_1),..,(λ_n,α_n) be the set of exhaustive and mutually exclusive uncertain events and let ϕ⊆Ω be an arbitrary event. Assume that π' is the revised possibility distribution computed from π using equation of Definition 1, namely $\forall\ \phi\subseteq\Omega$, $\Pi'(\phi)$=$max_{\lambda_i}(\alpha_i * \frac{\Pi(\phi,\lambda_i)}{\Pi(\lambda_i)})$. Then,
  By definition, $\forall\phi\subseteq\Omega$, $\Pi'(\phi)$=$max_{\lambda_i}(\alpha_i * \frac{\Pi(\phi,\lambda_i)}{\Pi(\lambda_i)})$. In particular,
  $\forall\lambda_j$, $\Pi'(\phi)$=$max_{\lambda_i}(\alpha_i * \frac{\Pi(\lambda_j,\lambda_i)}{\Pi(\lambda_i)})$.
  Recall that λ_1,..,λ_n are mutually exclusive, hence $\Pi(\lambda_j,\lambda_i)$=0 if i≠j. Then,
  $\forall\lambda_j$, $\Pi'(\phi)$=$\alpha_j * \frac{\Pi(\lambda_j)}{\Pi(\lambda_j)})$=$\alpha_j$.
  Let us now check that π' satisfies condition ii):

  By definition, we have $\Pi'(\phi|\lambda_i)$=$\frac{\Pi'(\phi,\lambda_i)}{\pi'(\lambda_i)}$.

  Applying Definition 1, $\Pi'(\phi|\lambda_i)$=$\frac{\alpha_i * \frac{\Pi(\phi,\lambda_i)}{\pi(\lambda_i)}}{\pi'(\lambda_i)}$.

  Since for i=1..n, $\Pi'(\lambda_i)$=α_i then $\pi'(\phi|\lambda_i)$=$\frac{\Pi(\phi,\lambda_i)}{\pi(\lambda_i)}$.

  Then, $\pi'(\phi|\lambda_i)$=$\pi(\phi|\lambda_i)$.

- Assume now that the possibility distribution π' satisfies conditions i) and ii) and let us retrieve the formula of Definition 1. By definition, $\forall\phi\in\Omega$,
  $\Pi'(\phi)$=$max_{i=1..n}(\Pi'(\phi,\lambda_i))$
  =$max_{i=1..n}(\pi'(\phi|\lambda_i) * \pi'(\lambda_i))$.
  Since π' satisfies i) and ii) then
  $\forall\phi\in\Omega$, $\Pi'(\phi)$=$max_{i=1..n}(\alpha_i * \pi(\phi|\lambda_i))$
  =$max_{i=1..n}(\alpha_i * \frac{\Pi(\phi,\lambda_i)}{\pi(\lambda_i)})$.

The following section analyzes the unicity of Jeffrey's rule for min-based conditioning.

Min-based possibilistic counterpart of Jeffrey's rule

The qualitative possibilistic setting uses the min-based conditioning defined as follows (Hisdal 1978) (we assume that $\Pi(\phi)\neq 0$), (see (Dubois & Prade 1988), (Yager 1982) for more details):

$$\pi_m(w_i|\phi) = \begin{cases} 1 & \text{if } \pi(w_i)\text{=}\Pi(\phi) \text{ and } w_i\in\phi; \\ \pi(w_i) & \text{if } \pi(w_i)< \Pi(\phi) \text{ and } w_i\in\phi; \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

Namely, conditioning π with a sure input ϕ consists in making impossible all the elements that are outside ϕ (interpretations that do not satisfy ϕ) and changing the possibility degree of the most plausible element of ϕ up to the value 1. Possibilistic belief revision based on the min-based possibilistic counterpart of Jeffrey's rule can be performed using the formula of Definition 2 proposed in (Dubois & Prade 1997).

Definition 2 Let π be a possibility distribution and λ_1,..,λ_n be a set of exhaustive and mutually exclusive events. Using the min-based conditioning of Equation 6, the revised possibility degree of any arbitrary event $\phi\subseteq\Omega$ is computed using the following formula:
$\forall\phi\subseteq\Omega$, $\Pi'(\phi)$=$max_{\lambda_i}(min(\Pi(\phi|\lambda_i),\pi'(\lambda_i))$.

Example 3 *Let us revise the joint possibility distribution given in Table 5 containing the joint possibility distribution encoding the doctor's initial beliefs. Let us now use the formula of Definition 2 in order to revise the joint possibility distribution of Table 5 in order to increase the possibility degree that an individual is smoker up to .9 (namely, $\pi'(S$=$True)$=.9 and $\pi'(S$=$False)$=1). The revised possibility distribution π' is given in Table 6. One*

S (Smoking)	C (Lung Cancer)	π(S,C)
True	True	0,4
True	False	0,8
False	True	0,5
False	False	1

Smoking	π(S)
True	0,8
False	1

π(C|S)
0,4
1,0
0,5
1,0

Table 5: Initial possibility distribution π.

S (Smoking)	C (Lung Cancer)	π'(S,C)
True	True	0,4
True	False	0,9
False	True	0,5
False	False	1,0

Smoking	π'(S)
True	0,9
False	1

π'(C|S)
0,4
1,0
0,5
1,0

Table 6: Revised possibility distribution π' corresponding to distribution π of Table 5 using Definition 2.

can check that in Tables 5 and 6, the conditional possibility $\pi'(S{=}True){=}.9$ and $\pi'(S{=}False){=}1$ and conditional possibility degrees $\pi(C|S)$ are exactly the same in π and π'.

From Definition 2, one can easily show that the revised possibility degree of any elementary event $w_j \in \Omega$ is computed following $Lemma$ 1.

Lemma 1

$$\forall w_j \in \lambda_i, \pi'(w_j) = \begin{cases} \alpha_i & \text{if } \pi(w_j){\geq}\alpha_i \text{ or } \pi(w_j){=}\Pi(\lambda_i); \\ \pi(w_j) & \text{otherwise.} \end{cases}$$

The formula of $Lemma$ 1 is an extension of the standard min-based conditioning of Equation 6 to the case where the input λ_i is uncertain. It states that the possibility degrees of the most plausible events in λ_i as well as the events that have possibility degrees at least equal to α_i become equal to α_i while the other elementary events of λ_i (namely those having possibility degrees less than λ_i) remain unchanged.
It is important to note that contrary to the probabilistic and quantitative possibilistic settings, there exist situations where equation of Definition 2 do not guarantee the existence of a solution satisfying condition i) and ii). Indeed, conditions i) and ii) cannot be completely satisfied if there exists some events λ_i whose new possibility degrees are decreased. The following example confirms this finding.

Example 4 *Assume now that the doctor wants to revise his beliefs in order to obtain $\pi'(S{=}True){=}.3$ and $\pi'(S{=}False){=}1$ (the doctor wants to decrease the possibility degree that an individual is smoker). The revised possibility distribution computed using equation of Definition 2 is given in Table 7. One can observe from Table 5 that*

S (Smoking)	C (Lung Cancer)	π'(S,C)
True	True	0,3
True	False	0,3
False	True	0,5
False	False	1,0

Smoking	π'(S)
True	0,3
False	1

π'(C|S)
1,0
1,0
0,5
1,0

Table 7: Revised possibility distribution π' corresponding to π of Table 5 using Definition 2.

$\pi(S{=}True|C{=}True){=}.4$ while
$\pi'(S{=}True|C{=}True){=}1$ in Table 7 violating condition ii).

The reason of this loss is due to the fact that when revising a possibility distribution using formula of Definition 2, imposing a possibility degree for a given uncertain event λ_i of $\pi'(\lambda_i){=}\alpha_i$ lesser than $\pi(\lambda_i)$ (namely, $\pi(\lambda_i){>}\alpha_i$), then there might be **several** elementary events $w_j{\in}\lambda_i$ (namely those having $\pi(w_j) > \alpha_i$) which will have their possibility degrees **downgraded** to α_i in the revised distribution π' (namely, $\pi'(w_j) = \alpha_i$). This results in **loosing the relative order of plausibility** of events w_j when moving from π to π' as in *Example 4*. In fact, assume that we have two elementary events w_1 and w_2 satisfying λ_i and $\pi(w_1){>}\pi(w_2)$. If $\Pi'(\lambda_i){<}\pi(w_1)$ and $\pi'(\lambda_i){<}\pi(w_2)$ then we can easily check that $\pi'(w_1){=}\pi'(w_2){=}\pi'(\lambda_i)$. It is clear in this case that even if w_1 and w_2 have different plausibility degrees in π, they will have the same plausibility in π'. As a consequence of this loss, conditional possibility degrees of events w_1 and w_2 given the uncertain events λ_i are not equal in π and π'. Hence, condition ii) cannot be satisfied in this case. However, when increasing the plausibility of event λ_i (namely, $\forall\lambda_i$, $\pi'(\lambda_i){\geq}\pi(\lambda_i)$) as in example of Table 6, the revision according to equation of Definition 2 will only upgrade the most plausible event belonging to λ_i in π to the new plausibility degree α_i. It is clear here that the relative order of events $w_j \in \lambda_i$ is preserved resulting in satisfying condition ii).
Let $|\lambda_i, \alpha_i, \pi|$ denote the number of different plausibility levels[3] (namely, different possibility degrees) where elementary events w_j satisfying the uncertain event λ_i are such that $\pi'(\lambda_i){=}\alpha_i{\leq}\pi(w_j)$. For instance, in example of Table 5 we have $|S{=}True,.9,\pi|{=}0$ while $|S{=}True,.3,\pi|{=}2$. Then, in the min-based possibilistic setting, when revising a possibility distribution π, there are two cases to be considered:

1. **Case 1:** If $\exists\lambda_i$, $|\lambda_i, \alpha_i, \pi|{\geq}2$ then the possibility distribution π' computed using equation of Definition 2 does not satisfy conditions i) and ii) as shown in *Example 4*. In fact, the following proposition shows that using Definition 2, there is no way to satisfy conditions i) and ii) if there exists an uncertain event λ_i where $|\lambda_i, \alpha_i, \pi|{\geq}2$.

   Proposition 2 Let π be a joint possibility distribution encoding the initial beliefs and (α_1, λ_1),..,(α_n, λ_n) be a set of uncertain inputs. Assume that there exists λ_i such that $|\lambda_i, \alpha_i, \pi|{\geq}2$. Then there is no revised possibility distribution π' that satisfies conditions i) and ii).

Proof of Proposition 2

Recall that $|\lambda_i, \alpha_i, \pi|{\geq}2$ means that there exists at least two elementary events w_1 and w_2 satisfying λ_i such that $\pi(w_1){\geq}\alpha_i$ and $\pi(w_2){\geq}\alpha_i$ and $\pi(w_1){\neq}\pi(w_2)$. Let w be an elementary event such that $\pi(w){=}\Pi(\lambda_i)$. Obviously, $\Pi(\lambda_i){>}\alpha_i$. Let w' be another elementary event such that $\pi(w){>}\pi(w'){\geq}\alpha_i$. By definition we have

$\Pi(w|\lambda_i){=}1$ and $\alpha_i{<}\Pi(w'|\lambda_i){=}\pi(w'){<}1$. (a)
Now, since π' satisfies condition i) we have:
$\Pi'(\lambda_i){=}\alpha_i$. (b)
The revised possibility distribution also satisfies condition ii) which means that:
$\Pi'(w|\lambda_i){=}\Pi(w|\lambda_i)$ and $\Pi'(w'|\lambda_i){=}\Pi(w'|\lambda_i)$. (c)
From (a) we have:
$\Pi'(w|\lambda_i){=}1$ which means that $\Pi'(w){=}\Pi'(\lambda_i){=}\alpha_i$. (d)
Similarly, from (a) we have:
$\alpha_i{<}\Pi'(w'|\lambda_i){=}\Pi'(w'){<}1$ which contradicts (d).

Proposition 2 states that when $|\lambda_i, \alpha_i, \pi|{\geq}2$, there is no hope to revise a possibility distribution in accordance with the two requirements i) and ii). It also shows that Definition 2 does not satisfy the two conditions i) and ii) in the situation where there is some input λ_i such that $|\lambda_i, \alpha_i, \pi|{\geq}2$. However, when $|\lambda_i, \alpha_i, \pi|{<}2$ there is a unique way to revise π (in accordance with i) and ii)) and corresponds to the possibility distribution computed using Definition 2.

[3]If several elementary events w_j have the same possibility degree, then they belong to a single plausibility level.

2. **Case 2:** If $\forall\lambda_i$, $|\lambda_i, \alpha_i, \pi|<2$, then the revision based on the min-based possibilistic counterpart of Jeffrey's rule accepts a unique solution given by the equation of Definition 2. Then we have the following propositions:

Proposition 3 Let π be a joint possibility distribution encoding the initial beliefs and (α_1, λ_1),..,(α_n, λ_n) be a set of uncertain, exhaustive and mutually exclusive events. If $\forall\lambda_i$, $|\lambda_i, \alpha_i, \pi|<2$ then the revised possibility distribution π', given by equation of Definition 2, satisfies conditions i) and ii).

Proof of Proposition 3

Assume that $|\lambda_i, \alpha_i, \pi|<2$. Let w be a model of λ_i such that $\pi(w)=\Pi(\lambda_i)$. We distinguish two cases:

(a) $\forall w' \in \lambda_i$, $\pi(w')=\pi(w)=\Pi(\lambda_i)$. In this case, $\forall w'$, $\Pi(w'|\lambda_i)=\Pi(w|\lambda_i)=1$.
Since π' satisfies ii) then $\forall w'$, $\Pi'(w'|\lambda_i)=\Pi(w|\lambda_i)=1$. Hence, $\forall w'$, $\Pi'(w')=\Pi'(\lambda_i)=\alpha_i$. This is exactly the one obtained by applying Definition 2.

(b) $\exists w'$ such that w' is a model of λ_i and $\beta=\pi(w')<\Pi(\lambda_i)$. In this case, $\Pi(w|\lambda_i)=1$ and $\Pi(w'|\lambda_i)=\pi(w')=\beta$. Since π' satisfies ii), we have:
$\Pi'(w|\lambda_i)=\Pi(w|\lambda_i)=1$ and $\Pi'(w'|\lambda_i)=\Pi(w'|\lambda_i)=\pi(w')$ and this also exactly corresponds to Definition 2.

The following proposition gives a stronger result. Namely when $\forall\lambda_i$, $|\lambda_i, \alpha_i, \pi|<2$, the revised possibility distribution π' is unique:

Proposition 4 Let π be a possibility distribution encoding the initial beliefs. If there exists a revised possibility distribution π' computed using Definition 2 and satisfying conditions i) and ii), then π' is unique.

Proof of Proposition 4
Let π' and π'' be two possibility distributions satisfying conditions i) and ii) and computed using formula of Definition 2. Then,

- $\forall\lambda_i, \Pi'(\lambda_i)=\alpha_i=\Pi''(\lambda_i)$ (condition (i)).
- $\forall\phi \subseteq \Omega$, $\Pi(\phi|\lambda_i)=\Pi'(\phi|\lambda_i)$ and $\Pi(\phi|\lambda_i)=\Pi''(\phi|\lambda_i)$ (condition (ii)).
- $\forall\phi \subseteq \Omega, \Pi'(\phi)=max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \Pi'(\lambda_i))$ and $\forall\phi, \Pi''(\phi)= max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \Pi''(\lambda_i))$ (from formula of Definition 2).

By definition, π' and π'' are different, which means that there exists an event $\phi \subseteq \Omega$ such that $\Pi'(\phi) \neq \Pi''(\phi)$. This implies that
$max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \pi'(\lambda_i)) \neq max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \pi''(\lambda_i))$
which also means that
$max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \alpha_i) \neq max_{\lambda_i}(min(\Pi(\lambda_i, \phi), \alpha_i)$
which is a contradiction.

The following example illustrates Case 1 and Case 2.

Example 5 *We continue using the Smoking/Lung cancer example. Let us in this example have three categories of smokers: Regular, Occasional and* Non $smoker$. *Assume that the initial beliefs are encoded by the joint possibility distribution* π *of Table 8. Let us now revise the joint possibility distribution* π *such that in the revised one* π' *we have* $\pi'(S{=}Regular){=}.5$, $\pi'(S{=}Occasional){=}.2$ *and* $\pi'(S{=}NonSmoker){=}1$. *The revised possibility distribution* π' *computed using the formula of Definition 2 is given in Table 9. In Table 9,*

S (Smoking)	C (Lung Cancer)	π(S,C)	π(C|S)
Regular	True	0,4	0,4
Regular	False	0,8	1,0
Occasional	True	0,3	0,3
Occasional	False	0,7	1,0
Non Smoker	True	0,5	0,5
Non Smoker	False	1	1,0

Smoking	π(S)
Regular	0,8
Occasional	0,7
Non Smoker	1

Table 8: Initial possibility distribution π.

S (Smoking)	C (Lung Cancer)	π'(S,C)	π'(C|S)
Regular	True	0,4	0,4
Regular	False	0,5	1,0
Occasional	True	0,2	1,0
Occasional	False	0,2	1,0
Non Smoker	True	0,5	0,5
Non Smoker	False	1,0	1,0

Smoking	π'(S)
Regular	0,5
Occasional	0,2
Non Smoker	1

Table 9: Revised possibility distribution π' corresponding to joint possibility distribution π of Table 8.

for all elementary events $w_j \in S{=}Regular$, $\pi'(w_j|S{=}Regular){=}\pi(w_j|S{=}Regular)$ *because* $|S{=}Regular, \alpha_i, \pi|{=}1$ *while in* $S{=}Occasional$, *we have* $\pi'(C{=}\ True|S{=}Occasional) \neq \pi(C{=}\ True|S{=}Occasional)$ *even if we have decreased the plausibility of event* $(S{=}Occasional, C{=}True)$ *because* $|S{=}Occasional, \alpha_i, \pi|{=}2$.

In order to summarize, we can say that the existence of a revised possibility distribution π' based on the min-based conditioning of Definition 2 and satisfying conditions i) and ii) is not guaranteed in case where there exists an uncertain event λ_i whose revision results in loosing the relative plausibility order of its own elementary events (namely, where $|\lambda_i, \alpha_i, \pi|{\geq}2$). However, if $|\lambda_i, \alpha_i, \pi|<2$, then there exists a unique solution π' given by Definition 2.

Conclusions

This paper addressed an issue related to belief revision in the framework of possibility theory. More precisely, we addressed the issue of the existence and unicity of the solution computed using the two possibilistic counterparts of Jeffrey's rule. As in the probabilistic framework, the product-based possibilistic counterpart of Jeffrey's rule accepts a unique solution. However, in the min-based setting, the possibilistic counterpart of Jeffrey's rule does not guarantee the existence of a solution satisfying the two conditions underlying Jeffrey's rule of conditioning. The problem with this rule is loosing the plausibility order of some elementary events after the revision task. The constraint imposed by Jeffrey's rule that the inputs must be completely accepted in the min-based possibilistic renders it impossible in some situations to satisfy the probability kinematics principle (condition ii)). A future work is to propose a weaker version of possibilistic counterpart of kinematics to be satisfied by the min-based conditioning.

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