Satisfiability Modulo Theories

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SMT combines subsolvers for certain classes of first order formulas with a DPLL SAT solver
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What is a first order formula?

Quantification (∃, ∀)
What is a first order formula?

Quantification ($\exists$, $\forall$)

Predicates: for all 2-colorings of the numbers from 1 to $n$ there exists an arithmetic progression of length $l$ among numbers of the same color.

$$\forall S \subset \{1...n\} (\exists S' \subset S P(S', l) \lor \exists S'' \subset \bar{S} P(S'', l))$$
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What is a first order theory?
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  Axioms, deduction rules
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  Theory: all axioms plus whatever can be deduced from them
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Theory: all axioms plus whatever can be deduced from them

Examples:

Real numbers: let $\mathcal{R}$ denote the set of real numbers

1. Operators $+$ and $\cdot$ exist, $\mathcal{R} + \mathcal{R} \rightarrow \mathcal{R}$, $\mathcal{R} \cdot \mathcal{R} \rightarrow \mathcal{R}$
2. $\{0, 1\} \subset \mathcal{R}$, $1 \cdot a \rightarrow a$, $0 + a \rightarrow a$
3. $a + b = b + a$, $a \cdot b = b \cdot a$
4. $a + (b + c) = (a + b) + c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
5. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
6. $\forall a \in \mathcal{R}$ $\exists -a, a^{-1} \in \mathcal{R}$ $a + (-a) = 0$, $a \cdot a^{-1} = 1$ ($a \neq 0$)
7. Operator $\geq$: $\forall x, y, z \in \mathcal{R}$ $x \geq y \rightarrow x + z \geq y + z$
   $\forall x, y \in \mathcal{R}$ $x \geq 0$ and $y \geq 0 \rightarrow x \cdot y \geq 0$
8. $LUB(S \subset \mathcal{R})$ exists if $S \neq \emptyset$, $S$ has upper bound
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More examples:

Theory of linear arithmetic \((ax + by \leq c)\)
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- Theory of bit vectors \((\text{concat}(bv_1, bv_2) == bv_3)\)
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More examples:

- Theory of linear arithmetic \( (ax + by \leq c) \)
- Theory of bit vectors \( (\text{concat}(bv_1, bv_2) = bv_3) \)
- Theory of arrays \( (arr[i := v_1][j] = v_2) \)
- Theory of uninterpreted functions \( (f(f(f(x))) = x) \)
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What can SMT do better than SAT?

- Boolean variables are replaced by predicates from various theories.
- The resulting language makes it easier to express properties.
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Boolean variables are replaced by predicates from various theories.
The resulting language makes it easier to express properties.

Examples: dataflow of words, as well as bits.

\( a < x < b \): simple tests involving three bit vectors.

but, as a collection of clauses:

\[
(x_d \land \neg a_d) \lor (\neg x_d \land \neg a_d \land x_{d-1} \land \neg a_{d-1}) \ldots
\]
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How does it work?

The SAT solver takes care of reasoning
When needed, it consults a theory solver which decides the validity of predicates.
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How to use it? Example: Yices

Formula example: \( i - 1 = j + 2, f(i - 3) \neq f(j + 6) \)
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Applications: equivalence checking, bounded model checking, test case generation, embedded in theorem provers
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Formula example: \( i - 1 = j + 2, f(i - 3) \neq f(j + 6) \)

Applications: equivalence checking, bounded model checking, test case generation, embedded in theorem provers

Example: yices ex1.ys

```plaintext
(define f::(-> int int))
(define i::int)
(define j::int)
(assert (= (- i 1) (+ j 2)))
(assert (/= (f (+ i 3)) (f (+ j 6)))))
```

Obviously false
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Example: yices ex2.ys

(define x::int)
(define y::int)
(define z::int)
(assert (= (+ (* 3 x) (* 6 y) z) 1))
(assert (= z 2))
(check)

Need the (check) to show unsatisfiable

$3 \times x + 6 \times y$ gives multiples of 3 ($3, 0, -3, -6, \ldots$)
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Example: yices -e ex3.ys

```
(define x::int)
(define y::int)
(define f::(-> int int))
(assert (/= (f (+ x 2)) (f (- y 1)))))
(assert (= x (- y 4)))
(check)
```

Counterexample: $x = 0, y = 4 \rightarrow f(2) = 1, f(3) = 5$
Example: yices -e ex4.ys

(define f::(-> int int))
(define i::int)
(define j::int)
(define k::int)
(assert+ (= (+ i (* 2 k)) 10))
(assert+ (= (- i 1) (+ j 2)))
(assert+ (= (f k) (f i)))
(assert+ (/= (f (+ i 3)) (f (+ j 6))))
(check)
(retract 2)
(check)

1. unsat core ids: 2 4
2. sat, \( i = 4, k = 3, j = 8, f(3) = 15, f(4) = 15, f(7) = 16, f(14) = 17 \)

Assertions made with \texttt{assert+} can be retracted.

Lemmas discovered in first check are reused in second.
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Example: yices -e ex5.ys

(define f::(-> int int))
(define i::int)
(define j::int)
(define k::int)
(assert+ (= (+ i (* 2 k)) 10) 10)
(assert+ (= (- i 1) (+ j 2)) 20)
(assert+ (= (f k) (f i)) 30)
(assert+ (/= (f (+ i 3)) (f (+ j 6))) 15)
(max-sat)

Returns:

sat
unsatisfied assertion ids: 4
(= i 4) (= k 3) (= j 1) (= (f 3) 8) (= (f 4) 8) (= (f 7) 9)
cost: 15
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Example: yices -e f1.ys

(define A1::(-> int int))
(define A2::(-> int int))
(define v::int) (define w::int)
(define x::int) (define y::int)
(define g::(-> (-> int int) int))
(define f::(-> int int))
(assert (= (update A1 (x) v) A2))
(assert (= (update A1 (y) w) A2))
(assert (/= (f x) (f y)))
(assert (/= (g A1) (g A2)))
(check)

unsat

Yices does not distinguish between functions and arrays
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Example: yices -e f1.ys

(define A1::(-> int int))
(define A2::(-> int int))
(define v::int) (define w::int)
(define x::int) (define y::int)
(define g::(-> (-> int int) int))
(define f::(-> int int))
(assert (= (update A1 (x) v) A2))
(assert (= (update A1 (y) w) A2))
(assert (/= (f x) (f y)))
(assert (/= (g A1) (g A2)))
(check)

unsat

Yices does not distinguish between functions and arrays
Remove (assert (/= (f x) (f y))))) to get

sat (= x 1) (= v 2) (= y 1) (= w 2) (= (A2 1) 2)
  (= (A1 1) 3) (= (g A1) 4) (= (g A2) 5)
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Example: yices -e f2.ys

(define f::(-> int int))
(assert (or (= f (lambda (x::int) 0))
          (= f (lambda (x::int) (+ x 1)))))
(define x::int)
(assert (and (>= x 1) (<= x 2)))
(assert (>= (f x) 3))
(check)

Returns:

sat
(= x 2) (= (f 2) 3) (= (f 4) 5)
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Example: yices -e dt.ys

(define-type list
    (datatype (cons car::int cdr::list) nil))
(define l1::list)
(define l2::list)
(assert (not (nil? l2)))
(assert (not (nil? l1)))
(assert (= (car l1) (car l2)))
(assert (= (cdr l1) (cdr l2)))
(assert (/= l1 l2))
(check)

unsat

so l1 and l2 must be the same!
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Example: yices -e bv.ys

```
(define b::(bitvector 4))
(assert (= b (bv-add 0b0010 0b0011)))
(check)
```

sat b=0b0101
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Example: yices -e d.ys

(define x::real)
(define y::int)
(define floor::(-> x::real
  (subtype (r::int) (and (>= x r) (< x (+ r 1)))))
(assert (and (> x 5) (< x 6)))
(assert (= y (floor x)))
(check)

sat (= x 11/2) (= y 5) (= (floor 11/2) 5)

State property of uninterpreted function
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Example: yices -e q.ys

(define f::(-> int int))
(define g::(-> int int))
(define a::int)
(assert (forall (x::int) (= (f x) x)))
(assert (forall (x::int) (= (g (g x)) x)))
(assert (/= (g (f (g a))) a))
(check)

unsat

Quantifier example