On aggregating binary relations using 0-1 integer linear programming

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Abstract

This paper is concerned with the general problem of aggregating many binary relations in order to find out a consensus. The theoretical background we rely on is the Relational Analysis (RA) approach. The latter method represents binary relations (BRs) as adjacency matrices, models relational properties as linear equations and finds a consensus by maximizing a majority-based criterion using 0-1 integer linear programming. Our contribution consists in a generalization of the theoretical results obtained in this framework. First, with regards to classical BRs on a single set, we provide new linear equations that correspond to relational properties that have not been covered yet such as semi-transitivity, quasi-transitivity or Ferrers relation. Second, we extend the BR aggregation problem to the case of maps or functions which are interpreted as BRs on two different sets. These results allow the RA approach to be a more general framework for dealing with BR aggregation problems. We also analyze the relationships between different BR aggregation problems and several tasks addressed in the artificial intelligence and machine learning fields. In that case, preference aggregation, clustering, and bi-clustering tasks are studied and we thus emphasize the underlying theoretical foundations of such problems from the point of view of BRs and their aggregations.

1 Introduction

Binary relations (BRs) are concepts which are commonly used in order to describe the relationships between several items. In everyday life for instance, "is the parent of" or "is married to", are examples of BRs that describe how two persons in the same family are linked. In many scientific fields, BRs are concepts that are ubiquitous. For example, in mathematics, functions are basically relations between two sets. We say that x is in relation with y according to the function f if $y = f(x)$. Another example is the database systems domain where a relational table typically codes BRs between objects and attributes.

In this paper, we assume that we are given many binary relations on the same set of items. As an example, we could consider a set of candidates to an election and the preference relations of many voters on those candidates. Then, the problem we want to address is to aggregate these individual BRs in order to find out a consensual binary relation (BR) that aims at summarizing them. Following the example mentioned beforehand, aggregating the voters' preference relation amounts to searching for a consensual preference relation on the candidates and determining the winner of the election. In the artificial intelligence field, quite similar situations appear in the context of multi-agent systems for example, where the preference relations of the agents on a set of alternatives are known and we want to aggregate them properly in order to achieve a decision. Another example is ensemble clustering in the context of unsupervised learning, where we want to aggregate the output of several clustering algorithms (that is to say their partitions) in order to obtain a more robust clustering of the data.

More formally, let us denote $\{R^1, R^2, \ldots, R^M\}$ the set of BRs of interest. In this work, we assume that each R^k could be of any type[1] although, in classical problems, we usually deal with a set of BRs of the same type. Then, we want to find a specified type of BR[2], R^*, such that:

$$R^* = \underset{R}{\text{Argmax}} \left(\text{Aggreg}(R^1, R^2, \ldots, R^M; R)\right) \quad (1)$$

where Aggreg is an aggregation procedure.

We mentioned previously two particular problems namely preference relations and equivalence relations (partitions) aggregation. The two latter cases are the most well-known BR aggregation problems. However, there are many other types of BRs and thus, many aggregation problems that fall under the general problem given in Equation (1).

In this paper, we propose a theoretical framework that tackles the BR aggregation problems from a general and flexible perspective. Our approach is based upon the Relational Analysis (RA) method originally introduced in the following papers (Michaud & Marcotorchino 1979; Marcotorchino & Michaud 1979; 1981; Marcotorchino 1987). In brief, this approach represents BRs through their adjacency matrices, models their relational properties using linear equations, aggregates them using the Condorcet procedure (Condorcet 1785), and finds a consensus by means of 0-1 integer linear programming (0-1 ILP).

[1]Different types of orders such as a mix of partial, strict linear orders and interval orders, for instance.

[2]Such as a strict linear order.

We propose a generalization which results in making the RA framework a unifying approach for modeling and optimally solving BR aggregation problems from a theoretical standpoint. Accordingly, our contribution is two-fold. First, we study relational properties that have not been covered yet in the RA method such as semi-transitivity, quasi-transitivity and Ferrers relation. Therefore, we show that more types of BRs can be taken into account through this approach such as semi- orders, interval orders. Second, we extend the RA approach in order to deal with BRs on two different sets. In that perspective, we show that maps which are interpreted from their general algebraic definition, can also be aggregated using the RA method.

The remainder of the paper is organized as follows. In section 2, we recall the basic definitions of BRs on one and two sets, their classical relational properties and some well-known types of BRs. In section 3, we recall some basics about the RA approach. Then, we provide linear equations that are equivalent to relational properties and we also extend the RA framework to BRs on two sets. In section 4, we recall the Condorcet aggregation procedure of boolean BRs and briefly consider multivalued BRs as well. Then, we show how to find consensual BRs using 0-1 ILP. In section 5, we analyze particular BR aggregation problems and establish their relationships with several machine learning tasks and related works. We finally illustrate our approach using an example in section 6 before concluding in section 7.

2 Basics on binary relations

In this section, we recall the definitions of BRs on one and two sets in subsections 2.1 and 2.2 respectively. We also remind the logical definition of well-known relational properties that a BR can satisfy. This leads us to the definition of special types of BRs such as preorders or interval orders for BRs on one set and partial maps and bijections in the case of BRs on two sets.

2.1 Binary relations on a single set

Definition 1. Binary relation on $\mathbb{A}$.

A binary relation R on a set of objects $\mathbb{A}$, is a couple $(\mathbb{A}, G(R))$, where $G(R)$ called the graph of the relation R, is a subset of the Cartesian product $\mathbb{A} \times \mathbb{A}$. If we have $(a, b) \in G(R)$, then we say that object a is in relation with object b for the relation R. This will be denoted by aRb.

Definition 2. Complement of a binary relation on $\mathbb{A}$.

Let $(\mathbb{A}, G(R))$ be a binary relation on the set $\mathbb{A}$. Then we can associate to R, its complement which is a binary relation denoted by $\overline{R}$ that is the subset of the Cartesian product $\mathbb{A} \times \mathbb{A}$ such that $(a, b) \in G(\overline{R}) \Leftrightarrow (a, b) \notin G(R)$.

There are different properties that a binary relation $(\mathbb{A}, G(R))$ can satisfy. Some of the most used of these relational properties are the following ones.

Property 1. Relational properties for $(\mathbb{A}, G(R))$.

1. Reflexive: $\forall a\,(aRa)$
2. Irreflexive: $\forall a\,\left(a\overline{R}a\right)$
3. Symmetric: $\forall a, b\,(aRb \Rightarrow bRa)$
4. Asymmetric: $\forall a \neq b\,\left(aRb \Rightarrow b\overline{R}a\right)$
5. Antisymmetric $(=)$: $\forall a, b\,((aRb \wedge bRa) \Rightarrow a = b)$
6. Antisymmetric $(\equiv)$: $\forall a, b\,((aRb \wedge bRa) \Rightarrow a \equiv b)$
7. Total (or complete): $\forall a \neq b\,(aRb \vee bRa)$
8. Transitive: $\forall a, b, c\,((aRb \wedge bRc) \Rightarrow aRc)$
9. Negatively transitive: $\forall a, b, c\,\left((a\overline{R}b \wedge b\overline{R}c) \Rightarrow a\overline{R}c\right)$
10. Semi-transitive: $\forall a, b, c, d\,((aRb \wedge bRc) \Rightarrow (aRd \vee dRc))$
11. Quasi-transitive: $\forall a, b, c\,\left((aRb \wedge b\overline{R}a \wedge bRc \wedge c\overline{R}b) \Rightarrow (aRc \wedge c\overline{R}a)\right)$
12. Acyclic: $\forall a, \ldots, s\,\left((aRb \wedge \ldots \wedge rRs) \Rightarrow a\overline{R}s\right)$
13. Ferrers: $\forall a, b, c, d\,((aRb \wedge cRd) \Rightarrow (aRd \vee cRb))$

The aforementioned notations $\forall a, b$ and $\forall a \neq b$ precisely means $\forall a, \forall b$ and $\forall a, \forall b (a \neq b)$ respectively. In this work we also distinguish two kinds of equality relations: $=$ and $\equiv$. The relation $a = b$ denotes the fact that a and b refer to exactly the same item of the set $\mathbb{A}$. On the contrary $a \equiv b$ means that a and b (which are not necessarily the same objects of $\mathbb{A}$) are indiscernible that is to say they have exactly the same properties. This concept is related to the identity of indiscernibles principle which was first introduced in (Leibniz 1686). In our context, the properties of an object a are related to its profile with respect to a relation R. The profile of a is in fact the set of objects $c \in \mathbb{A}$ such that aRc or cRa. Accordingly, the identity of indiscernibles principle can be formally expressed as follows:

$$\forall a, b\,(\forall c((aRc \Leftrightarrow bRc) \wedge (cRa \Leftrightarrow cRb)) \Rightarrow a \equiv b) \quad (2)$$

In order to better illustrate the difference between these two types of equivalence relations $=$ and $\equiv$, let us take the following example employing the antisymmetric property. Suppose on the one hand, that $\mathbb{A}$ is the set of real numbers and R is the "less or equal to" relation denoted $\leq$. In that case $a \leq b$ and $b \leq a$ implies $a = b$. On the other hand, let us take the following case: $\mathbb{A}$ is a set of candidates and R is the preference relation $\preceq$ such that $a \preceq b$ means that b is preferred or indifferent to a. In that case if a and b are two different candidates than $a \preceq b$ and $b \preceq a$ implies $a \equiv b$. More precisely this means that both candidates a and b are tied yet distinct.

Specific combinations of some relational properties listed in Property 1 lead to the definition of particular types of BRs. We have the following well-known cases.

Definition 3. Some types of binary relations on $\mathbb{A}$.

- A preorder is a BR that is reflexive and transitive.
- A partial order is a BR that is reflexive, antisymmetric and transitive.
- A total order or a linear order is a BR that is reflexive, antisymmetric, transitive and total.
- A strict total order or a strict linear order is a BR that is irreflexive, asymmetric, transitive and total.
- A semi-order is a BR that is reflexive, complete, Ferrers and semi-transitive.
- An interval order is a BR that is reflexive, complete and Ferrers.
- An equivalence relation is a BR that is reflexive, symmetric, and transitive.

2.2 Binary relations on two sets

In a more general way, we can define BRs on two different sets $\mathbb{A}$ and $\mathbb{B}$.

Definition 4. Binary relation on $\mathbb{A}$ and $\mathbb{B}$.
A binary relation R on two sets of objects $\mathbb{A}$ (the domain) and $\mathbb{B}$ (the codomain), is a triple $(\mathbb{A}, \mathbb{B}, G(R))$, where $G(R)$ called the graph of the relation R, is a subset of the Cartesian product $\mathbb{A} \times \mathbb{B}$. If we have $(a, \boldsymbol{\alpha}) \in G(R)$, then we say that object a is in relation with object $\boldsymbol{\alpha}$ for the relation R. This will be denoted by $aR\boldsymbol{\alpha}$.

Notice that, in the sequel, we will always use Latin letters for items in $\mathbb{A}$ and bold Greek letters for items in $\mathbb{B}$.

We can also define the complement of a binary relation $(\mathbb{A}, \mathbb{B}, G(R))$ which is similar to Definition 2. The classical relational properties of BRs on two sets are given in Property 2.

Property 2. Relational properties for $(\mathbb{A}, \mathbb{B}, G(R))$.

1. Left-total: $\forall a\,(\exists \boldsymbol{\alpha}(aR\boldsymbol{\alpha}))$
2. Right-total: $\forall \boldsymbol{\alpha}\,(\exists a(aR\boldsymbol{\alpha}))$
3. Surjective: Left-total and right-total
4. Injective ($=$): $\forall a, b, \forall \boldsymbol{\alpha}\,((aR\boldsymbol{\alpha} \wedge bR\boldsymbol{\alpha}) \Rightarrow a = b)$
5. Injective ($\equiv$): $\forall a, b, \forall \boldsymbol{\alpha}\,((aR\boldsymbol{\alpha} \wedge bR\boldsymbol{\alpha}) \Rightarrow a \equiv b)$
6. Functional ($=$): $\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \forall a\,((aR\boldsymbol{\alpha} \wedge aR\boldsymbol{\beta}) \Rightarrow \boldsymbol{\alpha} = \boldsymbol{\beta})$
7. Functional ($\equiv$): $\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \forall a\,((aR\boldsymbol{\alpha} \wedge aR\boldsymbol{\beta}) \Rightarrow \boldsymbol{\alpha} \equiv \boldsymbol{\beta})$
8. Bijective ($=$,$=$)[3]: Surjective, injective ($=$), functional ($=$)
9. Bijective ($=$,$\equiv$): Surjective, injective ($=$), functional ($\equiv$)
10. Bijective ($\equiv$,$=$): Surjective, injective ($\equiv$), functional ($=$)
11. Bijective ($\equiv$,$\equiv$): Surjective, injective ($\equiv$), functional ($\equiv$)

As previously, we consider two kinds of equality relations $=$ and $\equiv$. However, since $\mathbb{A}$ and $\mathbb{B}$ are basically two different sets, these relations concern the comparison of internal objects of $\mathbb{A}$ and $\mathbb{B}$. As a result, in that case, the identity of indiscernibles principle is rather formulated as follows:

$$\forall a, b\,(\forall \boldsymbol{\alpha}(aR\boldsymbol{\alpha} \Leftrightarrow bR\boldsymbol{\alpha}) \Rightarrow a \equiv b) \tag{3}$$
$$\forall \boldsymbol{\alpha}, \boldsymbol{\beta}\,(\forall a(aR\boldsymbol{\alpha} \Leftrightarrow aR\boldsymbol{\beta}) \Rightarrow \boldsymbol{\alpha} \equiv \boldsymbol{\beta}) \tag{4}$$

We can then recall the definition of the following particular types of BRs on two sets.

Definition 5. Some types of binary relations on $\mathbb{A}$ and $\mathbb{B}$.

- A partial map is a BR that is functional.
- A map is a BR that is left-total and functional.
- An injective map is a BR that is left-total, functional and injective.
- A surjective map is a BR that is functional and surjective.
- A bijective map is a BR that is functional, subjective and injective.

[3] The first equality sign corresponds to the one used for the injective property while the second one, is the one used for the functional property.

3 Representing binary relations using relational matrices

The Relational Analysis (RA) method is a flexible approach for dealing with BR aggregation problems. In this section we begin by introducing relational matrices that represent BRs using adjacency matrices. Moreover, we show how we can express relational properties that we have recalled previously by means of linear equations. In subsection 3.1, we deal with BRs on one single set whereas in subsection 3.2 we focus on BRs on two sets. In that perspective, we point out our contributions which consist in, on the one hand, giving the linear equations of relational properties of BRs on one set that have not been covered so far and, on the other hand, extending the RA method to the case of BRs on two sets.

3.1 Binary relations on a single set

In the RA framework, a BR R on a set $\mathbb{A}$ is represented by its adjacency matrix which is a binary pairwise matrix. In our context, we rather call such adjacency matrices, relational matrices, since the latter satisfy particular properties that we introduce in what follows.

We denote C the relational matrix associated to the binary relation R. Its general term is given as follows, $\forall a, b$:

$$C_{ab} = \begin{cases} 1 & \text{if } aRb \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Following Definition 2, we can similarly define, $\overline{C}$, the relational matrix of the complement of R. It is related to C through the following relation, $\forall a, b$:

$$\overline{C}_{ab} = 1 - C_{ab} \tag{6}$$

In Proposition 1, we give linear equations in terms of C that are equivalent to the logical definitions of the relational properties recalled in Property 1. These linear constraints are fundamental from a modeling standpoint since they allow one to use 0-1 ILP solvers to optimally solve BR aggregation problems.

Proposition 1. Relational properties of $(\mathbb{A}, G(\mathcal{R}))$ as linear equations using the RA formalism.

1. Reflexive: $\forall a; C_{aa} = 1$
2. Irreflexive: $\forall a; C_{aa} = 0$
3. Symmetric: $\forall a, b; C_{ab} = C_{ba}$
4. Asymmetric: $\forall a \neq b; C_{ab} + C_{ba} \leq 1$
5. Antisymmetric ($=$): $\forall a \neq b; C_{ab} + C_{ba} \leq 1$
6. Antisymmetric ($\equiv$): $\forall a, b;$
$$\forall c; \begin{cases} C_{ab} + C_{ba} + C_{ac} - C_{bc} \leq 2 \\ C_{ab} + C_{ba} - C_{ac} + C_{bc} \leq 2 \\ C_{ab} + C_{ba} + C_{ca} - C_{cb} \leq 2 \\ C_{ab} + C_{ba} - C_{ca} + C_{cb} \leq 2 \end{cases}$$
7. Total (or complete): $\forall a \neq b; C_{ab} + C_{ba} \geq 1$
8. Transitive: $\forall a, b, c; C_{ab} + C_{bc} - C_{ac} \leq 1$
9. Negatively transitive: $\forall a, b, c; C_{ac} \leq C_{ab} + C_{bc}$
10. Semi-transitive: $\forall a, b, c, d; C_{ab} + C_{bc} - C_{ad} - C_{dc} \leq 1$

11. Quasi-transitive: $\forall a, b, c$;
$\begin{cases} C_{ab} - C_{ba} + C_{bc} - C_{cb} - C_{ac} \leq 1 \\ C_{ab} - C_{ba} + C_{bc} - C_{cb} + C_{ca} \leq 2 \end{cases}$
12. Acyclic: $\forall a, \dots, s; C_{ab} + \dots + C_{rs} + C_{as} \leq S - 1$ with S being the length of the sequence $a, \dots, s$.
13. Ferrers: $\forall a, b, c, d; C_{ab} + C_{cd} - C_{ad} - C_{cb} \leq 1$

In Proposition 1, the linear equations 1, 2, 3, 4, 7, 8 were already given in (Michaud & Marcotorchino 1979). As a result, to our knowledge the other linear equations are new results. Particularly, the properties 6, 10, 11, 13 are non-trivial. We give a sketch of their respective proof in appendix. Furthermore, our approach that consists in distinguishing two different kinds of equivalence relations $=$ and $\equiv$ and which leads to different linear equations is also new.

Because of these linear equations, relational matrices are adjacency matrices of a certain kind. Below, we give two examples that illustrate the particular structure that characterizes relational matrices. We assume $\mathbb{A} = \{a, b, c, d\}$. First, we define the following equivalence relation: $R^1 = \{(a,a); (b,b); (c,c); (d,d); (a,b); (b,a); (a,c); (c,a); (b,c); (c,b)\}$. Second, we set $R^2 = \{(a,a); (b,b); (c,c); (d,d); (a,b); (a,c); (a,d); (b,a); (b,c); (b,d); (c,d)\}$ which, with regards to the properties it satisfies, represents a total order with antisymmetry[4] ($\equiv$). The respective relational matrices C^1 and C^2 are the following ones:

$$C^1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad ; \quad C^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} a & b & c & d \end{matrix} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

3.2 Binary relations on two sets

In the case of BRs on two sets, the relational matrix C related to the BR R is defined by, $\forall a, \forall \boldsymbol{\alpha}$:

$$C_{a\boldsymbol{\alpha}} = \begin{cases} 1 & \text{if } aR\boldsymbol{\alpha} \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

Similarly to Equation (6) we can also define the relational matrix $\overline{C}$ of the BR R. We give in Proposition 2 the linear equations in terms of C that are equivalent to the logical definitions of the relational properties given in Property 2.

Proposition 2. Relational properties as linear equations using the RA formalism for $(\mathbb{A}, \mathbb{B}, G(\mathcal{R}))$ binary relations.

1. Left-total: $\forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} \geq 1$
2. Right-total: $\forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \geq 1$
3. Surjective: $\begin{cases} \forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} \geq 1 \\ \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \geq 1 \end{cases}$
4. Injective ($=$): $\forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \leq 1$
5. Injective ($\equiv$): $\forall a, b, \forall \boldsymbol{\alpha}$;
$\forall \boldsymbol{\beta}; \begin{cases} C_{a\boldsymbol{\alpha}} + C_{b\boldsymbol{\alpha}} + C_{a\boldsymbol{\beta}} - C_{b\boldsymbol{\beta}} \leq 2 \\ C_{a\boldsymbol{\alpha}} + C_{b\boldsymbol{\alpha}} - C_{a\boldsymbol{\beta}} + C_{b\boldsymbol{\beta}} \leq 2 \end{cases}$
6. Functional ($=$): $\forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} \leq 1$
7. Functional ($\equiv$): $\forall \boldsymbol{\alpha}, \boldsymbol{\beta}, \forall a$;
$\forall b; \begin{cases} C_{a\boldsymbol{\alpha}} + C_{a\boldsymbol{\beta}} + C_{b\boldsymbol{\alpha}} - C_{b\boldsymbol{\beta}} \leq 2 \\ C_{a\boldsymbol{\alpha}} + C_{a\boldsymbol{\beta}} - C_{b\boldsymbol{\alpha}} + C_{b\boldsymbol{\beta}} \leq 2 \end{cases}$
8. Bijective (=,=): $\begin{cases} \forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} = 1 \\ \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} = 1 \end{cases}$
9. Bijective (=,$\equiv$): $\begin{cases} \forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} \geq 1 \\ \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} = 1 \end{cases}$
10. Bijective ($\equiv$,=): $\begin{cases} \forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} = 1 \\ \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \geq 1 \end{cases}$
11. Bijective ($\equiv$,$\equiv$): $\begin{cases} \forall a; \sum_{\boldsymbol{\alpha} \in \mathbb{B}} C_{a\boldsymbol{\alpha}} \geq 1 \\ \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \geq 1 \\ \forall a, b; \forall \boldsymbol{\alpha}, \boldsymbol{\beta}; \\ \begin{cases} C_{a\boldsymbol{\alpha}} + C_{a\boldsymbol{\beta}} + C_{b\boldsymbol{\alpha}} - C_{b\boldsymbol{\beta}} \leq 2 \\ C_{a\boldsymbol{\alpha}} + C_{a\boldsymbol{\beta}} - C_{b\boldsymbol{\alpha}} + C_{b\boldsymbol{\beta}} \leq 2 \\ C_{a\boldsymbol{\alpha}} - C_{a\boldsymbol{\beta}} + C_{b\boldsymbol{\alpha}} + C_{b\boldsymbol{\beta}} \leq 2 \end{cases} \end{cases}$

The linear equations related to the bijective ($\equiv$,$\equiv$) property was introduced in another context in (Marcotorchino 1987). Except this case, to our knowledge, this work is the first attempt to represent relational properties of BRs on two sets as linear equations in a unifying framework. We give a sketch of their respective proof in appendix.

We illustrate the following types of BRs on $\mathbb{A} = \{a, b, c\}$ and $\mathbb{B} = \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta}\}$. On the one hand, we consider the bijective (=,$\equiv$) map $R^3 = \{(a, \boldsymbol{\alpha}); (b, \boldsymbol{\beta}); (b, \boldsymbol{\delta}); (c, \boldsymbol{\gamma})\}$. On the other hand, we define the relational matrix of the bijective ($\equiv$, $\equiv$) map, $R^4 = \{(a, \boldsymbol{\alpha}); (a, \boldsymbol{\beta}); (b, \boldsymbol{\alpha}); (b, \boldsymbol{\beta}); (c, \boldsymbol{\gamma}); (c, \boldsymbol{\delta})\}$. These BRs are represented by the relational matrices below:

$$C^3 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \; ; \; C^4 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{matrix} \boldsymbol{\alpha} & \boldsymbol{\beta} & \boldsymbol{\gamma} & \boldsymbol{\delta} \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

4 Aggregating binary relations

In the previous section, we have recalled the definition of relational matrices and more particularly we have provided the linear equations that are equivalent to the relational properties introduced in section 2. Thereby, we have stated new contributions which contribute to extend the scope of applications of the RA approach in modeling BRs. In this section, we deal with the aggregation procedure of BRs in the RA framework. We recall the aggregation of many BRs into collective relational matrices in subsection 4.1. Next, in subsection 4.2, we introduce majority-based criteria that we want to maximize in order to determine a consensual BR of a special type. In subsection 4.3, we briefly consider the more general case of multivalued BRs. Then, we explain in subsection 4.4 how we can optimally solve BRs aggregation problems using 0-1 integer linear programming.

Without loss of generality, throughout this section, we suppose that we deal with BRs on two sets $\mathbb{A}$ and $\mathbb{B}$. But, this case generalizes the case of BRs on a single set, if we assume $\mathbb{A} = \mathbb{B}$ and $a = \boldsymbol{\alpha}$, $b = \boldsymbol{\beta}$ and so on.

[4] Note that regarding C^2, objects a and b are such that $a \neq b$ but $a \equiv b$.

4.1 Collective relational matrices

Finding a general way to aggregate BRs is not straightforward. For example, let us assume that we have a set of linear orders as individual BRs such as in the context of preference aggregation. We could represent the preferences linearly as an ordered list of the items of interest. $a \preceq_1 b \preceq_1 c$ is an example of such a representation. Thereby, if we take another linear order such as $b \preceq_2 c \preceq_2 a$, one can see that it is not easy to define an aggregate of both linear orders from these representations.

It turns out that by representing BRs by means of their relational matrices, we can define a simple way to aggregate those pieces of information. This aggregation procedure is simply the sum over the relational matrices and it naturally works for any kinds of BRs (over the same set(s) of items). We call the resulting matrix, the positive collective relational matrix which is denoted by $\mathbf{C}$. If we assume M BRs $R^k; k = 1, \ldots, M$ and $C^k = k = 1, \ldots, M$ being their respective relational matrices then, we have:

$$\mathbf{C} = \sum_{k=1}^{M} C^k \tag{8}$$

The general term of $\mathbf{C}_{a\boldsymbol{\alpha}}$ is the number of relations that support "a is in relation with $\boldsymbol{\alpha}$" and thus, $\forall a, \forall \boldsymbol{\alpha}; \mathbf{C}_{a\boldsymbol{\alpha}} \in \{0, 1, \ldots, M\}$. Similarly to the positive collective relational matrix $\mathbf{C}$, we can define its complement, the negative collective relational matrix, $\overline{\mathbf{C}}$, which is the following sum:

$$\overline{\mathbf{C}} = \sum_{k=1}^{M} \overline{C}^k \tag{9}$$

In that case, $\overline{\mathbf{C}}_{a\boldsymbol{\alpha}}$ is the number of relations that support "a is not in relation with $\boldsymbol{\alpha}$". We have the following equation that links both collective matrices, $\forall a, \forall \boldsymbol{\alpha}$:

$$\mathbf{C}_{a\boldsymbol{\alpha}} + \overline{\mathbf{C}}_{a\boldsymbol{\alpha}} = M \tag{10}$$

4.2 Majority-based criteria

From both collective matrices $\mathbf{C}$ and $\overline{\mathbf{C}}$ we can consider the pairwise "difference" matrix, D, given by $\mathbf{C} - \overline{\mathbf{C}}$. Therefore, $D_{a\boldsymbol{\alpha}}$ compares the number of (positive) BRs that support "a is in relation with $\boldsymbol{\alpha}$" to the number of (negative) BRs that supports "a is not in relation with $\boldsymbol{\alpha}$". We then have $\forall a, \forall \boldsymbol{\alpha}; D_{a\boldsymbol{\alpha}} \in \{-M, \ldots, -1, 0, 1, \ldots, M\}$. We can see that the greater $D_{a\boldsymbol{\alpha}}$, the more we should consider to put a in relation with $\boldsymbol{\alpha}$ if we want to promote the consensus and likewise, the lower $D_{a\boldsymbol{\alpha}}$, the more we should consider to not put a in relation with $\boldsymbol{\alpha}$.

As a result, the Condorcet criterion aims at maximizing the following function F in order to find a consensus:

$$\max_X F(C^1, \ldots, C^M; X) = \sum_{(a,\boldsymbol{\alpha}) \in \mathbb{A} \times \mathbb{B}} \underbrace{\left(\mathbf{C}_{a\boldsymbol{\alpha}} - \overline{\mathbf{C}}_{a\boldsymbol{\alpha}}\right)}_{D_{a\boldsymbol{\alpha}}} X_{a\boldsymbol{\alpha}}$$

where X is the relational matrix representing R^* the consensual BR we are looking for.

We can simplify the previous function in order to underline the notion of simple majority. Indeed, the aforementioned equation is equivalent to $\max_X \sum_{(a,\boldsymbol{\alpha})} \left(\mathbf{C}_{a\boldsymbol{\alpha}} - \frac{M}{2}\right) X_{a\boldsymbol{\alpha}}$. In the latter expression we can see that the Condorcet criterion amounts to comparing the number of BRs that support a relation between a pair of objects to the simple majority $\frac{M}{2}$.

As a generalization of such types of consensus criterion, we can consider the following majority-based criteria:

$$\max_X F(C^1, \ldots, C^M, m; X) = \sum_{(a,b) \in \mathbb{A} \times \mathbb{B}} \underbrace{(\mathbf{C}_{a\boldsymbol{\alpha}} - m)}_{D_{a\boldsymbol{\alpha}}} X_{a\boldsymbol{\alpha}}$$

where m is a parameter that belongs to $[0, M]$ and which represents the specified majority. This parameter could also be dependent on each pair $(a, \boldsymbol{\alpha})$[5].

4.3 The more general case of multivalued binary relations

The material presented previously focuses on boolean BRs aggregation problems. A more general case, we could consider is the one for which we are given fuzzy BRs. In that case, $\forall k; \forall a, \forall \boldsymbol{\alpha}; C^k_{a\boldsymbol{\alpha}}, \overline{C}^k_{a\boldsymbol{\alpha}} \in [0, 1]$.

As far far as the aggregation procedure of such BRs is concerned, there are in the literature, plenty of aggregation operators which could be used in order to determine fuzzy collective relational matrices. Power means, quasi-arithmetic means or triangular norms and co-norms are examples of such operations (see for example, (Calvo *et al.* 2002)). We can thus formulate the positive and negative fuzzy collective relational matrices as follows, $\forall a, \forall \boldsymbol{\alpha}$:

$$\begin{aligned} \mathbf{C}_{a\boldsymbol{\alpha}} &= A(C^1_{a\boldsymbol{\alpha}}, C^2_{a\boldsymbol{\alpha}}, \ldots, C^M_{a\boldsymbol{\alpha}}) \\ \overline{\mathbf{C}}_{a\boldsymbol{\alpha}} &= A(\overline{C}^1_{a\boldsymbol{\alpha}}, \overline{C}^2_{a\boldsymbol{\alpha}}, \ldots, \overline{C}^M_{a\boldsymbol{\alpha}}) \end{aligned}$$

where A is an aggregation operators.
As a consequence, we can define a "difference" matrix D given by $\mathbf{C} - \overline{\mathbf{C}}$.

Finally, an even more general setting is when we are given a "difference" matrix D such that $\forall a, \forall \boldsymbol{\alpha}; -\infty < D_{a\boldsymbol{\alpha}} < +\infty$ and the greater the positive value of $D_{a\boldsymbol{\alpha}}$ the more we should promote aR^*b and the lower the negative value of $D_{a\boldsymbol{\alpha}}$ the less we should promote aR^*b. All criteria we have presented so far are particular cases of the following objective function:

$$\max_X F(D; X) = \sum_{(a,\boldsymbol{\alpha}) \in \mathbb{A} \times \mathbb{B}} D_{a\boldsymbol{\alpha}} X_{a\boldsymbol{\alpha}} \tag{11}$$

4.4 Finding a consensus using 0-1 ILP

On the one hand, we have shown that the basic relational properties that define special types of BRs such as interval orders and bijective maps among others, can be expressed as linear equations in terms of the relational matrix. On the other hand, we have previously defined objective criteria F which are linear with respect to the relational matrix X that represents the consensual BR R^* we want to determine. As a consequence, we can use 0-1 ILP to model and optimally solve any of the BR aggregation problems that fall under the material described in this paper. To this end, one could apply the following general procedure:

[5] See for example (Ah-Pine 2009) in the context of clustering.

1. We start by finding positive and negative collective relational matrices by summing (or aggregating depending on the nature of the BRs) the relational matrices of the individual BRs. Note that the set of individual BRs could be heterogeneous meaning that one could consider the aggregation of a mix of several types of BR.
2. Then, we maximize the objective function F which general formulation is given in Equation (11) with respect to X using 0-1 ILP and to specify the type of BR we would like to obtain, we constrain X to satisfy the corresponding linear constraints given by Propositions 1 or 2.

It is worth mentioning that the described approach allows one to integrate other simple constraints such as "must-link" or "cannot-link" relations. In that context we assume that we have prior knowledge about some pairs of items for which we know that they should or should not be in relation. Thus, if we want the pair of objects $(a, \boldsymbol{\alpha})$ to be in relation then we add the constraint $X_{a\boldsymbol{\alpha}} = 1$. On the contrary, if according to some expertise, we know that those objects must not be in relation, we impose $X_{a\boldsymbol{\alpha}} = 0$. As a result, it is easy to model constrained BR aggregation problems in the RA framework.

5 Some related applications and works

We have presented a theoretical framework for modeling various BR aggregation problems by means of 0-1 ILP. In this section, we analyze different related applications and works. To that regard, we recall that our contribution is at the theoretical level. Therefore, it is not the purpose of this paper to design methods that tackle BR relation problems with large sets of items. Indeed, BR aggregation problems are combinatorial and NP-Hard problems (Wakabayashi 1998; Hudry 2008). Besides, our approach aims at optimally solving those problems which means that we cannot avoid the combinatorial aspect of such problems using our framework. Thus, in practice, the 0-1 ILP approach could only be applied when the sizes of $\mathbb{A}$ and $\mathbb{B}$ are not large but regardless of the number of individual BRs.

Accordingly, for the tasks we are going to present, there are, in the literature, numerous heuristics that target large datasets and are designed in the goal of finding an approximate solution to the corresponding BR aggregation problem in a reasonable amount of time. We will not focus on such contributions but rather cite papers that are interested in analyzing BR aggregation problems from the modeling and combinatorial optimization viewpoints.

In subsection 5.1 we are concerned with the preference aggregation problems and in a more general manner order relations aggregation problems. Then, in subsection 5.2, we focus on clustering problems where the consensual BR that we want to obtain is a hard partition or an equivalence relation. Finally in subsection 5.3, we are interested in BRs on two sets and we show that assignment and bi-clustering problems can both be seen as the search for a consensual bijection but of two different kinds.

5.1 Preference and order relations aggregation

Preference aggregation problems are encountered in various domains. For example, in economics and more particularly in voting and social choice theories, aggregating preference relations such as linear orders among a set of candidates is a core problem (Condorcet 1785; Arrow 1963; Kemeny & Snell 1972; Fishburn 1972; Michaud & Marcotorchino 1979; Barthélémy & Monjardet 1991).

Ordinal data analysis and multicriteria decision making are other fields that also study related problems (Marcotorchino & Michaud 1979; Lerman 1981; Figueira, Mousseau, & Roy 2005; Oztürk, Tsoukiàs, & Vincke 2005).

More recently, in the artificial intelligence field there have been numerous papers related to agent's preference relations aggregation problems and in diverse contexts (Chevaleyre *et al.* 2007; Pini *et al.* 2008; Conitzer, Rognlie, & Xia 2009; Shoham & Leyton-Brown 2009). Lately, there have been some papers interested in the aggregation of interval orders (Öztürk & Tsoukiàs 2006; Berre, Marquis, & Öztürk 2009). With regards to our contribution, no such 0-1 ILP formulation of this problem as the one we design in this paper, has been proposed so far to our knowledge.

Apart from the applications mentioned previously, in the information retrieval domain we can also notice many recent papers about rank aggregation for meta-search problems that are also related to this topic (Montague & Aslam 2002; Farah & Vanderpooten 2007). In the latter application, the aim is to combine different search engines results in order to determine a more robust top list of relevant results.

Finally, as far as the complexity and optimization viewpoints of linear orders aggregation are concerned, we can cite the following papers (Grötschel, Jünger, & Reinelt 1984; Reinelt 1985; Barthélémy, Guenoche, & Hudry 1989; Wakabayashi 1998).

5.2 Clustering

In unsupervised learning, one main task consists in clustering a set of objects departing from a feature matrix that describes the latter in an euclidean space. The goal is to find a partition such that objects within a cluster are highly similar. There are many fields and numerous papers that have been studying similarity measures and clustering techniques and heuristics (see for example (Mirkin 1996)) since this topic is an important one in data analysis.

However, (Zahn 1964; Régnier 1965; Mirkin 1974) are some papers that originally addressed the clustering problem from the BR point of view. In that context, the work presented in (Michaud & Marcotorchino 1979; Barthélémy & Monjardet 1991) and lately in (Ailon, Charikar, & Newman 2008), contributed to unify the preference aggregation and the clustering problems (also known as median relation problems).

Following (Michaud & Marcotorchino 1979; Marcotorchino 1986), the clustering problem of categorical data can be interpreted as an equivalence relations (partition) aggregation problem and be optimally solved by using 0-1 ILP. More recently, there has been a renewed interest in that type of work in the context of correlation clustering (Bansal, Blum, & Chawla 2004; Demaine & Immorlica 2003; Joachims & Hopcroft 2005). In that case, the problem is to find a partition departing from an undirected graph of the items of interest.

Finally as mentioned earlier, there have also been many papers in cluster aggregation techniques or ensemble clustering which consist in combining several clustering outputs in order to compute a more robust partition (Ailon, Charikar, & Newman 2008; Gionis, Mannila, & Panayiotis 2007).

5.3 Assignment problems and Bi-clustering

A classical assignment problem aims at finding a relation that assigns each element of $\mathbb{A}$ to a unique element of $\mathbb{B}$ such that it minimizes the cost (or maximizes the profit) in a weighted bipartite graph. Is is easy to see that this problem amounts to searching for a bijective (=,=) map given by relation 8 in Proposition 2. Thus, our results allow such a problem to be embedded in a more general framework. First, the distinction between $=$ and $\equiv$ allows one to design different related problems by using distinct kinds of bijections. Second, the assignment problem can be seen as a particular case of BR aggregation problems.

Another related problem is the bi-clustering one also known as co-clustering or two-mode clustering in the literature (Mirkin 1996; Anagnostopoulos, Dasgupta, & Kumar 2008). In such a problem the goal is to simultaneously find a clustering of the rows and the columns of a matrix. Therefore, the output is a set of non-overlapping bi-clusters where a cluster of elements of $\mathbb{A}$ is associated to a unique cluster of elements of $\mathbb{B}$. Such a decomposition is of interest in many domains such as text-mining (Dhillon, Mallela, & Modha 2003) or microarray data analysis (Pensa & Boulicaut 2008). In fact, we can see this problem as the search for a bijective ($\equiv$,$\equiv$) map given by relation 11 in Proposition 2. This binary relation is aimed at maximizing an objective function dependent on a "difference" matrix D such that the higher $D_{a\boldsymbol{\alpha}}$ the greater we should consider to put a and $\boldsymbol{\alpha}$ in relation that is to say into the same bi-cluster[6]. The relation 11 was already proposed in (Marcotorchino 1987). However, the interpretation was not in terms of BRs but in terms of block seriation. Indeed, in that case, the relation 11 was called "impossible triad" whereas we have shown that they are related to the injective ($\equiv$) and functional ($\equiv$) relational properties.

6 An illustrative example

As a proof of concept, let us take the following example employing three different kinds of order relations, C^1, C^2, C^3, given as follows:

$$C^1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} \quad ; \quad C^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$C^3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

[6] R^4 and its relational matrix C^4 introduced in subsection 3.2 gives an example of two bi-clusters.

C^1 is a complete preorder[7], C^2 a total linear order and C^3 a partial order. Let us assume moreover, that there are three BRs of each of those three kinds of order relation so that we want to aggregate nine individual BRs. Using Equations (8) and (9), we obtain the related positive and negative collective relational matrices $\mathbf{C}$ and $\overline{\mathbf{C}}$ respectively. Thereby, we can compute the following "difference" matrix $D = \mathbf{C} - \overline{\mathbf{C}}$:

$$D = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 9 & 3 & -3 & 3 \\ 3 & 9 & 3 & 3 \\ -3 & -9 & 9 & 9 \\ -9 & -9 & -3 & 9 \end{pmatrix} \end{array}$$

We can use our framework in order to fit D with different types of BR. As a first example, if we want to find an interval order that sums up as good as possible the individual BRs than, we can optimally determine such a BR by solving the following 0-1 ILP:

$$\begin{array}{c} \max_X \sum_{(a,b)\in\mathbb{A}^2} D_{ab}X_{ab} \\ \\ \text{subject to:} \\ \forall a, b; X_{ab} \in \{0,1\} \\ \forall a; X_{aa} = 1 \text{ (Reflexivity)} \\ \forall a \neq b; X_{ab} + X_{ba} \geq 1 \text{ (Total)} \\ \forall a,b,c,d; X_{ab} + X_{cd} - X_{ad} - X_{cb} \leq 1 \text{ (Ferrers)} \end{array} \qquad (12)$$

We used the open source GLPK package[8] (GNU Linear Programming Kit) to solve the previous 0-1 integer linear problem. We obtain the following solution X^1, with an optimal criterion value equal to 57:

$$X^1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

If instead of an interval order relation, we want to find a strict linear order, then we replace the linear constraints in the Problem (12) accordingly and we obtain the following relational matrix X^2, which represents the strict linear order $b \succ c \succ a \succ d$. The corresponding optimal criterion value in that case is 18.

$$X^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

If the BR we are searching for is a preorder thus in that case, the criterion value we obtain is 57 and the relational matrix is the following one:

$$X^3 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

[7] Equivalent to a total linear order with antisymmetry ($\equiv$), in that case.

[8] http://www.gnu.org/software/glpk/

Note that in the latter case, the solution is not forced to be complete and it is actually optimal to not put in relation a and c or c and a. On the contrary, in the former case, X^2 is a strict linear order and by definition, it is a total relation. One can notice that having $X^2(a,c) = 1$ instead of $X^2(c,a) = 1$, would give another optimal strict linear order. Indeed, we have $D(a,c) = D(c,a) = -3$ and the latter shift would not change the type of the BR since X^2 would remain asymmetric, total and transitive.

Finally, since our method is a flexible one, we could also consider to fit D with an equivalence relation. Using again a 0-1 ILP solver, the optimal solution found gives a criterion value of 48 and the related relational matrix is as follows:

$$X^4 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

7 Conclusion

In this paper, we are interested in the modeling aspect of BR aggregation problems. Our main contribution is the extension of the RA approach which leads to the design of a unifying framework for modeling and optimally solving BR aggregation problems by means of 0-1 ILP. Our approach encompasses the aggregation problem of the most important BRs on a single set such as linear orders, semi-orders, interval orders, equivalence relations and also extend the method to the case of BRs on two sets such as maps. Besides, it also enables the aggregation problem of mixed BRs to be tackled in a flexible manner, by allowing one to aggregate many kinds of BRs and specify any type of BR as a consensual BR.

Appendix

Proof. Proof of Proposition 1.
The relations 1, 2, 3, 4 and 7 are straightforward while 8 which is related to the transitivity property is given in (Michaud & Marcotorchino 1979). Notice that the negative transitivity 9 is obtained by replacing into 8, the terms with their complements. Then, we can see that relation 5 reduces to relation 4 despite the fact that they are originally defined in different ways. To prove the antisymmetry (=) 5, we use the contrapositive of the logical definition 5 given in Property 1:

5. $\forall a, b \left(a \neq b \Rightarrow a\overline{R}b \vee b\overline{R}a\right)$
$\Leftrightarrow \forall a \neq b; \overline{C}_{ab} + \overline{C}_{ba} \geq 1$
$\Leftrightarrow \forall a \neq b; 1 - C_{ab} + 1 - C_{ba} \geq 1$
$\Leftrightarrow \forall a \neq b; C_{ab} + C_{ba} \leq 1$

To prove relation 6 related to the antisymmetry (≡), we prove that the premises $(aRb \wedge bRa)$ imply that $\forall c\left((aRc \Leftrightarrow bRc) \wedge (cRa \Leftrightarrow cRb)\right)$ which implies $a \equiv b$ according to Equation (2). To this end, let us denote the four linear equations given in 6 as follows:

6. $\forall c; \begin{cases} C_{ab} + C_{ba} + C_{ac} - C_{bc} \leq 2 & i \\ C_{ab} + C_{ba} - C_{ac} + C_{bc} \leq 2 & ii \\ C_{ab} + C_{ba} + C_{ca} - C_{cb} \leq 2 & iii \\ C_{ab} + C_{ba} - C_{ca} + C_{cb} \leq 2 & iv \end{cases}$

From the premises, we get $C_{ab} = C_{ba} = 1$ and thus $C_{ab} + C_{ba} = 2$. It remains four other variables $C_{ac}, C_{bc}, C_{ca}, C_{cb}$ but which are employed by pairs (C_{ac}, C_{bc}) in i and ii and (C_{ca}, C_{cb}) in the remaining cases. We can draw the two following tables that enumerate all possible combinations of truth values:

C_{ac}	C_{bc}	i	ii
1	1	2	2
1	0	3	1
0	1	1	3
0	0	2	2

and

C_{ca}	C_{cb}	iii	iv
1	1	2	2
1	0	3	1
0	1	1	3
0	0	2	2

From these tables we can see that the cases for which the four linear equations i, ii, iii and iv are simultaneously less are equal than 2 are exactly the ones for which $(C_{ac} = C_{bc})$ and $(C_{ca} = C_{cb})$ (first and fourth rows of the tables) which is equivalent to $(aRc \Leftrightarrow bRc) \wedge (cRa \Leftrightarrow cRb)$.
The same technique can be applied to prove the other remaining relations. In brief, we infer from the premises the corresponding truth values associated to the terms of relational matrices. Next, we identify the other terms and enumerate each possible combination of truth values. We replace any complement such as $\overline{C}_{ab}$ (if any) with $1 - C_{ab}$. We then compute the value of the linear equations for all possible combinations. We finally show that the cases that respect the given linear constraints are the ones that exactly satisfy the logical definition of the property. □

Proof. Proof of Proposition 2.
The relations 1, 2, 3 are straightforward. The proof of the injective (=) property given by relation 4 starts similarly to the proof of the relation 5 in Proposition 1 provided previously.

4. $\forall a, b, \forall \boldsymbol{\alpha} \left(a \neq b \Rightarrow (a\overline{R}\boldsymbol{\alpha} \vee b\overline{R}\boldsymbol{\alpha})\right)$
$\Leftrightarrow \forall a \neq b; \forall \boldsymbol{\alpha}; \overline{C}_{a\boldsymbol{\alpha}} + \overline{C}_{b\boldsymbol{\alpha}} \geq 1$
$\Leftrightarrow \forall a \neq b; \forall \boldsymbol{\alpha}; 1 - C_{a\boldsymbol{\alpha}} + 1 - C_{b\boldsymbol{\alpha}} \geq 1$
$\Leftrightarrow \forall a \neq b; \forall \boldsymbol{\alpha}; C_{a\boldsymbol{\alpha}} + C_{b\boldsymbol{\alpha}} \leq 1$
$\Leftrightarrow \forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} \leq 1$

The proof of relation 6 is the same as the proof of relation 4. Then we can show relations 5 and 7 using the technique we introduced at the end of Proof of Proposition 1. Relation 8 is a straightforward consequence of linear equations 3, 4 and 6. With regards to relation 9 which is related to bijections (=,≡), we get from linear equations 3 and 4: $\forall \boldsymbol{\alpha}; \sum_{a \in \mathbb{A}} C_{a\boldsymbol{\alpha}} = 1$. The latter linear constraint implies relation 7 which therefore, becomes redundant. The same goes for relation 10. Finally, relation 11, is the bijection (≡,≡) and it is the conjunction between linear constraints given in 3, 5 and 7. Since the two last relations have one linear constraint in common, this explains why the conjunction of 5 and 7 reduces to three linear constraints. □

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